

Spin Geometry on Quantum Groups via Covariant Differential Calculi*

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Abstract

Let \mathcal{A} be a cosemisimple Hopf $*$ -algebra with antipode S and let Γ be a left-covariant first order differential $*$ -calculus over \mathcal{A} such that Γ is self-dual (see Section 2) and invariant under the Hopf algebra automorphism S^2 . A quantum Clifford algebra $\text{Cl}(\Gamma, \sigma, g)$ is introduced which acts on Woronowicz' external algebra Γ^\wedge . A minimal left ideal of $\text{Cl}(\Gamma, \sigma, g)$ which is an \mathcal{A} -bimodule is called a spinor module. Metrics on spinor modules are investigated. The usual notion of a linear left connection on Γ is extended to quantum Clifford algebras and also to spinor modules. The corresponding Dirac operator and connection Laplacian are defined. For the quantum group $\text{SL}_q(2)$ and its bicovariant $4D_\pm$ -calculi these concepts are studied in detail. A generalization of Bochner's theorem is given. All invariant differential operators over a given spinor module are determined. The eigenvalues of the Dirac operator are computed.

Key Words: quantum groups, covariant differential calculus, spin geometry

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0 Introduction

Noncommutative geometry has been invented in the eighties as a new field by the pioneering work of A. Connes [4]. Nowadays it is commonly expected that

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it will provide new mathematical tools and methods for applications in theoretical physics (quantum gravity, physics at small distances). On the other hand, quantum groups also arose in the eighties as new classes of noncommutative non-cocommutative Hopf algebras [10]. A general framework for bicovariant noncommutative differential calculi over Hopf algebras has been provided by S. L. Woronowicz [21]. In the meantime a deep algebraic theory of such calculi has been developed (see, for instance, [16], Chapter 14). Unfortunately, the bicovariant differential calculus on quantum groups does not fit into the realm of the noncommutative geometry of A. Connes. As a first step to connect both theories one needs spin structures and Dirac operators on quantum groups. In this paper we study whether covariant differential calculi permit the setup of a spin geometry. The bridge to A. Connes' noncommutative geometry will be considered in a forthcoming paper.

The theory of spin and related structures for (pseudo)riemannian manifolds has its origin in Dirac's relativistic theory of the electron (1928). Comprehensive introductions into the subject can be found (for instance) in the books by E. Cartan [3], A. Crumeyrolle [5], and H. B. Lawson and M.-L. Michelsohn [17]. Recently some attempts have been made to generalize these structures to noncommutative algebras. Connections on bimodules of differential one-forms have been examined by several authors [6], [9], [2], [7], [13]. In the paper of R. Bautista et. al. [1] quantum Clifford algebras are constructed in the case where the braiding on the tensor product of one-forms satisfies the Hecke relation.

The aim of this paper is to present generalizations of certain well-known structures of the spin geometry to left-covariant differential $*$ -calculi on quantum groups.

Throughout we deal with a cosemisimple Hopf $*$ -algebra \mathcal{A} and with a self-dual S^2 -invariant left-covariant differential $*$ -calculus over \mathcal{A} . A definition of the corresponding quantum Clifford algebra and its main involution is given. It is proved that the quantum Clifford algebra has a representation on Woronowicz' external algebra. Spinor modules are defined as left ideals and \mathcal{A} -subbimodules of the quantum Clifford algebra. Conditions for the extension of a connection on the differential calculus to the latter objects are given. The connection Laplacian and the Dirac operators are constructed. The symmetry of the Dirac operator with respect to metrics on spinor modules is analyzed. After a discussion of the general situation the quantum group $SL_q(2)$ (with three non-isomorphic $*$ -structures) and the two 4-dimensional bicovariant differential calculi on it are studied in detail.

It is proved that there exists a minimal spinor module \mathcal{S}_0 of the quantum Clifford algebra. We introduce a Hopf $*$ -algebra $\tilde{\mathcal{A}}$ which contains $\mathcal{A} = \mathcal{O}(\mathrm{SL}_q(2))$ as a Hopf subalgebra. Then one can define a right coaction of $\tilde{\mathcal{A}}$ on \mathcal{S}_0 which is compatible with the Clifford multiplication and the original right coaction of the differential calculus. Therefore $\tilde{\mathcal{A}}$ can be interpreted as the function algebra of a covering of the quantum group $\mathrm{SL}_q(2)$. Also the uniqueness of the metric on \mathcal{S}_0 is proved. It is important to emphasize that this metric is hermitean and non-degenerate but it is not positive definite. Connections on the quantum Clifford algebra and on \mathcal{S}_0 are found. It turns out that with our definitions there exists no torsion free connection on the differential one-forms. Finally all invariant differential operators on the spinor module \mathcal{S}_0 are determined. The eigenvalues of the Dirac operator are computed and a generalization of Bochner's theorem is proved.

The paper is organized in the following way. In Section 1 the definition and some properties of cosemisimple Hopf $*$ -algebras are given. In Section 2 we collect the assumptions on the differential calculus over the Hopf $*$ -algebra. In Section 3 we introduce and study quantum Clifford algebras, spinor modules and metrics on it. In Section 4 we investigate linear connections on differential forms, quantum exterior and quantum Clifford algebras and on spinor modules. We introduce a duality of left connections which makes the definition of the connection Laplacian possible. In Section 5 the Dirac operator is defined and some of its properties are proved. In Section 6 we apply the general theory developed in the preceding sections to the 4-dimensional bicovariant differential calculi on the quantum groups $\mathrm{SU}_q(2)$, $\mathrm{SU}_q(1, 1)$ and $\mathrm{SL}_q(2, \mathbb{R})$. All structures are worked out in great detail. Quantum spin groups for the differential structures on these quantum groups and metrics and connections on the spinor module \mathcal{S}_0 are determined. In Section 7 we use generalized ℓ -functionals and Clebsch-Gordan coefficients to obtain all invariant differential operators on the spinor module \mathcal{S}_0 and on one of its subbimodules \mathcal{S}_0^+ . One of our main results (Theorem 7.2) states that the algebra of invariant differential operators on \mathcal{S}_0^+ is commutative and generated by two elements and one relation (92). In Section 7 we also compute the eigenvalues of the Dirac operator and prove a generalization of Bochner's theorem (Theorem 7.5). In two appendices some properties of the Hopf algebras $\mathcal{O}(\mathrm{SL}_q(2))$ and $U_q(\mathfrak{sl}_2)$ and a corresponding graphical calculus for the morphism spaces are recalled.

Let us fix some notation. If not otherwise stated, we follow the definitions and

conventions of the monograph [16]. In this paper \mathcal{A} denotes a cosemisimple Hopf $*$ -algebra over the complex numbers with coproduct Δ , counit ε and antipode S . We use Sweedler's notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$. The symbol $\text{Mor}(v, w)$ denotes the vector space of intertwiners T of corepresentations v and w of \mathcal{A} . Throughout we use Einstein's convention to sum over repeated indices.

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1 Cosemisimple Hopf $*$ -algebras

A Hopf algebra \mathcal{A} is called a Hopf $*$ -algebra if \mathcal{A} is a $*$ -algebra with involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$ such that $\Delta(a^*) = (* \otimes *)\Delta(a)$ for any $a \in \mathcal{A}$. A Hopf algebra \mathcal{A} is called cosemisimple, if any corepresentation of \mathcal{A} is a direct sum of simple corepresentations. If not otherwise stated, \mathcal{A} always denotes a cosemisimple Hopf $*$ -algebra.

One of the most important properties of cosemisimple Hopf algebras is the existence of a unique left- and right-invariant functional (the Haar functional) h on \mathcal{A} such that $h(1) = 1$. Moreover, the Haar functional is left- and right-regular, i. e. $b \in \mathcal{A}$ and $h(ab) = 0$ ($h(ba) = 0$) for any $a \in \mathcal{A}$ implies $b = 0$.

Since the functional h' on \mathcal{A} , defined by $h'(a) := \overline{h(a^*)}$ for all $a \in \mathcal{A}$ is also left- and right-invariant and $h'(1) = 1$, we have $h' = h$ and therefore $h(a^*) = \overline{h(a)}$ for any $a \in \mathcal{A}$.

It is well known that there is an automorphism ρ of \mathcal{A} such that $h(ab) = h(b\rho(a))$ for any $a, b \in \mathcal{A}$. More precisely, if $w = (w_{ij})$ is a simple corepresentation of \mathcal{A} and $F = (F_{kl}) \in \text{Mor}(w, w^{\text{cc}})$, $F \neq 0$, then F is an invertible matrix with complex entries, $\text{Tr} F \neq 0$, $\text{Tr} F^{-1} \neq 0$ and

$$\rho(w_{kl}) = \frac{F_{km} F_{nl} \text{Tr} F^{-1}}{\text{Tr} F} w_{mn}. \quad (1)$$

Let us define a mapping $\beta : \mathcal{A} \rightarrow \mathcal{A}$ by $\beta(a) := \rho(a)^*$. Then we have $h(ab) = h(b\beta(a)^*)$ for any $a, b \in \mathcal{A}$ and β becomes an algebra involution of \mathcal{A} . Indeed, β is an antilinear antiendomorphism of \mathcal{A} and

$$h(ab) = h(b\beta(a)^*) = \overline{h(\beta(a)b^*)} = \overline{h(b^*\beta^2(a)^*)} = h(\beta^2(a)b) \quad (2)$$

for any $a, b \in \mathcal{A}$. Hence $a = \beta^2(a)$ for any $a \in \mathcal{A}$.

Lemma 1.1. *For the algebra involution β of \mathcal{A} the following formula holds:*

$$\Delta(\beta(a)) = \beta(a_{(1)}) \otimes S^2(a_{(2)}^*) \quad \text{for all } a \in \mathcal{A}. \quad (3)$$

Proof. Both sides of (3) are antilinear mappings from \mathcal{A} to $\mathcal{A} \otimes \mathcal{A}$. Hence it suffices to prove the lemma for $a = w_{ij}$, where w_{kl} are matrix elements of an irreducible corepresentation of \mathcal{A} such that $\Delta(w_{ij}) = w_{ik} \otimes w_{kj}$. Let F be an invertible morphism of the corepresentations w and w^{cc} . For the left hand side of (3) we obtain

$$\Delta(\beta(w_{ij})) = \Delta(\rho(w_{ij})^*) = \Delta\left(\frac{\overline{F_{ik}} \overline{F_{lj}} \overline{\text{Tr} F^{-1}}}{\overline{\text{Tr} F}} w_{kl}^*\right)$$

by (1). Since Δ is a $*$ -homomorphism, the latter is equal to

$$\begin{aligned} \frac{\overline{F_{ik}} \overline{F_{lj}} \overline{\text{Tr} F^{-1}}}{\overline{\text{Tr} F}} w_{km}^* \otimes w_{ml}^* &= \frac{\overline{F_{ik}} \overline{F_{mn}} \overline{\text{Tr} F^{-1}}}{\overline{\text{Tr} F}} w_{km}^* \otimes \overline{F_{nr}^{-1}} \overline{F_{lj}} w_{rl}^* \\ &= \beta(w_{in}) \otimes (F_{nr}^{-1} w_{rl} F_{lj})^*. \end{aligned} \quad (*)$$

Finally, $F \in \text{Mor}(w, w^{\text{cc}})$ gives $w_{rl} F_{lj} = S^{-2}(S^2(w_{rl}) F_{lj}) = S^{-2}(F_{rl} w_{lj})$. Hence $(F_{nr}^{-1} w_{rl} F_{lj})^* = (F_{nr}^{-1} F_{rl} S^{-2}(w_{lj}))^* = S^2(w_{nj}^*)$ and (3) follows from (*). \blacksquare

2 S^2 -invariant differential calculi

Let \mathcal{A} be a cosemisimple Hopf $*$ -algebra and (Γ, d) a left-covariant first order differential $*$ -calculus over \mathcal{A} . Let \mathcal{X} denote the quantum tangent space of Γ . Since S^2 is a Hopf algebra automorphism of \mathcal{A} , there is a left-covariant first order differential calculus (Γ', d') with quantum tangent space $\mathcal{X}' := S^2(\mathcal{X})$ (see [11]). We call (Γ, d) *S^2 -invariant*, if (Γ, d) and (Γ', d') are isomorphic, i.e. $\mathcal{X}' = \mathcal{X}$. **In this article all first order differential calculi are assumed to be S^2 -invariant left-covariant differential $*$ -calculi.**

Let $\{X_i \mid i = 1, \dots, n = \dim \mathcal{X}\}$ be a basis of \mathcal{X} and let $\{\theta_i \mid i = 1, \dots, n\}$ be the dual basis of the vector space of left-invariant 1-forms Γ_L . This means that for the differential mapping $d : \mathcal{A} \rightarrow \Gamma$ the formula

$$da = a_{(1)} X_i(a_{(2)}) \theta_i, \quad a \in \mathcal{A}, \quad (4)$$

holds. Recall that $X_i \in \mathcal{A}^\circ$ and there exist functionals $f_j^i \in \mathcal{A}^\circ$ such that $\Delta X_i = \varepsilon \otimes X_i + X_j \otimes f_j^i$. Moreover, the formula $\theta_i a = a_{(1)} f_j^i(a_{(2)}) \theta_j$ holds for $a \in$

\mathcal{A} and $i = 1, \dots, n$. If $S^2(X_i) = F_j^i X_j$ for some $F_j^i \in \mathbb{C}$, then the mapping $\theta_i \mapsto S^2(\theta_i) := F_j^i \theta_j$ extends uniquely to an isomorphism of the left-covariant \mathcal{A} -bimodule Γ and we have $S^2(da) = dS^2(a)$ for all $a \in \mathcal{A}$.

Recall that the involution $*$ of \mathcal{A}° is defined by

$$f^*(a) = \overline{f(S(a)^*)} \quad \text{or} \quad f(a^*) = \overline{S(f)^*(a)}. \quad (5)$$

Since (Γ, d) is a $*$ -calculus, we have $\mathcal{X}^* = \mathcal{X}$. Hence there is a matrix $E = (E_j^i)$ such that $X_i^* = E_j^i X_j$ (and therefore $\theta_i^* = -\theta_j \overline{E_i^j}$) for $i = 1, \dots, n$. Let now $B = (B_j^i)_{i,j=1,\dots,n}$ denote the complex matrix $B = \bar{E}^T E$. This implies that

$$S^2(X_i)^* = B_j^i X_j. \quad (6)$$

Proposition 2.1. *The setting $\beta(\theta_i) := \overline{B_i^j} \theta_j$, $\beta(a)$ as in the previous section, defines an involution of the \mathcal{A} -bimodule Γ . Moreover, $\beta(da) = -d\beta(a)$ for any $a \in \mathcal{A}$.*

Proof. Since X_i is an element of the Hopf algebra \mathcal{A}° , from $S(S(X_i)^*)^* = X_i$ we obtain $S^2(S^2(X_i)^*)^* = X_i$ for any $i = 1, \dots, n$. Therefore

$$X_i = S^2(B_j^i X_j)^* = \overline{B_i^j} S^2(X_j)^* = \overline{B_i^j} B_k^j X_k \quad (7)$$

and so $\overline{B} B = \text{id}$. Hence $\beta^2(\theta_i) = \beta(\overline{B_i^j} \theta_j) = B_i^j \overline{B_j^k} \theta_k = \theta_i$ for any $i = 1, \dots, n$. Now we only have to show that β is well-defined, that is $0 = \beta(a)\beta(\theta_i) - \beta(\theta_j)\beta(a_{(1)}f_j^i(a_{(2)}))$ (which is formally the image under β of the element $\theta_i a - a_{(1)}f_j^i(a_{(2)})\theta_j$) for $a \in \mathcal{A}$ and $i = 1, \dots, n$. For this we compute

$$\begin{aligned} \Delta(S^2(X_i)^*) &= ((S^2 \otimes S^2)\Delta(X_i))^* = 1 \otimes S^2(X_i)^* + S^2(X_j)^* \otimes S^2(f_j^i)^* \\ &= 1 \otimes B_j^i X_j + B_k^j X_k \otimes S^2(f_j^i)^*, \end{aligned} \quad (8)$$

$$\Delta(B_j^i X_j) = B_j^i 1 \otimes X_j + B_j^i X_k \otimes f_j^k. \quad (9)$$

Hence we obtain $B_j^i f_j^k = B_k^j S^2(f_j^i)^*$, i. e. $f_l^k = \overline{B_i^l} B_j^i f_j^k = \overline{B_i^l} B_k^j S^2(f_j^i)^*$. Therefore

$$\begin{aligned} \beta(\theta_j)\beta(a_{(1)}f_j^i(a_{(2)})) &= \overline{B_j^k} \theta_k \beta(a_{(1)}) \overline{\overline{B_m^j} B_i^n S^2(f_m^n)^*(a_{(2)})} \\ &= \overline{B_j^k} \beta(a_{(1)}) f_l^k (S^2(a_{(2)}^*)) \theta_l B_m^j \overline{B_i^n} S^2(f_m^n)(S^{-1}(a_{(3)}^*)) \\ &= \overline{B_i^n} \beta(a_{(1)})(S^2(f_l^k)S(f_m^n))(a_{(2)}^*) \theta_l = \overline{B_i^n} \beta(a_{(1)}) \delta_l^n \overline{\varepsilon(a_{(2)})} \theta_l = \beta(a) \overline{B_i^l} \theta_l. \end{aligned}$$

Here from the 1st to the 2nd line we used (3) and (5). Similarly one can compute $\beta(da)$.

$$\begin{aligned}
\beta(a_{(1)}X_i(a_{(2)})\theta_i) &= \beta(\theta_i)\beta(a_{(1)})\overline{X_i(a_{(2)})} = \overline{B_i^j}\theta_j\beta(a_{(1)})\overline{X_i(a_{(2)})} \\
&= \overline{B_i^j}\beta(a_{(1)})f_k^j(S^2(a_{(2)}^*))\theta_k S^{-2}(X_i^*)(S(a_{(3)}^*)) \\
&= \overline{B_i^j}\beta(a_{(1)})(B_l^i X_l S(f_k^j))(S(a_{(2)}^*))\theta_k = \beta(a_{(1)})S(X_j S(f_k^j))(a_{(2)}^*)\theta_k \\
&= \beta(a_{(1)})(-S^2(X_k))(a_{(2)}^*)\theta_k = -\beta(a_{(1)})X_k(S^2(a_{(2)}^*))\theta_k = -d\beta(a).
\end{aligned}$$

■

If (Γ, d) is a bicovariant FODC over \mathcal{A} then there is a canonical method (given by S. L. Woronowicz [21]) to construct a differential calculus (Γ^\wedge, d) over \mathcal{A} with first order part (Γ, d) . This construction is based on the existence of a braiding, an automorphism σ of the bicovariant \mathcal{A} -bimodule $\Gamma \otimes_{\mathcal{A}} \Gamma$ satisfying the braid relation. The mapping σ is defined by the formula

$$\sigma(\rho \otimes_{\mathcal{A}} \rho') = \rho' \otimes_{\mathcal{A}} \rho, \quad \rho \in \Gamma_L, \rho' \in \Gamma_R. \quad (10)$$

However, for the existence of a braiding bicovariance is not necessary (for an example see [11]). Let m_\wedge denote the canonical mapping $m_\wedge : \Gamma^{\otimes_{\mathcal{A}} 2} \rightarrow \Gamma^{\wedge 2} = \Gamma^{\otimes_{\mathcal{A}} 2} / \ker(\text{id} - \sigma)$.

Let (Γ, d) be a $(S^2$ -invariant left-covariant) first order differential $*$ -calculus over \mathcal{A} . Assume that there is an automorphism σ of the left-covariant \mathcal{A} -bimodule $\Gamma \otimes_{\mathcal{A}} \Gamma$ such that

1. σ satisfies the braid relation $\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}$ on $\Gamma^{\otimes_{\mathcal{A}} 3}$,
2. $\ker(\text{id} - \sigma)$ contains all elements $\omega(a_{(1)}) \otimes_{\mathcal{A}} \omega(a_{(2)}) \in \Gamma \otimes_{\mathcal{A}} \Gamma$, where $a \in \mathcal{R}_\Gamma = \{b \in \mathcal{A} \mid \omega(b) = 0, \varepsilon(b) = 0\}$,
3. $S^2(\sigma\rho_2) = \sigma(S^2(\rho_2))$ for any $\rho_2 \in \Gamma^{\otimes_{\mathcal{A}} 2}$,
(where $S^2(\rho \otimes_{\mathcal{A}} \rho') = S^2(\rho) \otimes_{\mathcal{A}} S^2(\rho')$, $\rho, \rho' \in \Gamma$),
4. $(\sigma\rho_2)^* = \sigma(\rho_2^*)$ or $(\sigma\rho_2)^* = \sigma^{-1}(\rho_2^*)$ for any $\rho_2 \in \Gamma^{\otimes_{\mathcal{A}} 2}$.

Such a mapping σ is called a *braiding of Γ* . Now for any $k \in \mathbb{N}_0$ there is an antisymmetrizer A_k , defined by σ [21], which is an automorphisms of the left-covariant \mathcal{A} -bimodule $\Gamma^{\otimes_{\mathcal{A}} k} = \Gamma \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Gamma$ (k times). In particular, we have $A_1 = \text{id}$ and $A_2 = \text{id} - \sigma$. Moreover, the direct sum of the kernels of

the antisymmetrizer A_k is a twosided ideal in $\bigoplus_{k=0}^{\infty} \Gamma^{\otimes_{\mathcal{A}} k}$. If we now define $\Gamma^{\wedge k} := \Gamma^{\otimes_{\mathcal{A}} k} / \ker A_k$, $\Gamma^{\wedge} := \bigoplus_{k=0}^{\infty} \Gamma^{\wedge k}$, then there is a canonical differential calculus (Γ^{\wedge}, d) over \mathcal{A} with first order part (Γ, d) .

Let us call a FODC (Γ, d) *self-dual*, if there exists a braiding σ of Γ and a left-covariant σ -metric $g : \Gamma \otimes_{\mathcal{A}} \Gamma \rightarrow \mathcal{A}$ [12]. More exactly, the mapping g should be a homomorphism of left-covariant \mathcal{A} -bimodules, be non-degenerate, and should satisfy the equations $g\sigma = g$ on $\Gamma^{\otimes_{\mathcal{A}} 2}$ and $g_{12}\sigma_{23}^{\pm}\sigma_{12}^{\pm} = g_{23}$ on $\Gamma^{\otimes_{\mathcal{A}} 3}$. Further, for compatibility with the involution we assume that

$$g(\rho \otimes_{\mathcal{A}} \rho')^* = g(\rho'^* \otimes_{\mathcal{A}} \rho^*) \quad \text{for } \rho, \rho' \in \Gamma. \quad (11)$$

Lemma 2.2. *Let Γ be a bicovariant \mathcal{A} -bimodule with canonical braiding σ . Suppose that there exists a homomorphism $g : \Gamma \otimes_{\mathcal{A}} \Gamma \rightarrow \mathcal{A}$ of bicovariant \mathcal{A} -bimodules. Then the equation $g\sigma = g$ is fulfilled if and only if*

$$g(S^2(\rho') \otimes_{\mathcal{A}} \rho) = g(\rho \otimes_{\mathcal{A}} \rho'), \quad \rho, \rho' \in \Gamma_L. \quad (12)$$

Proof. First recall that $S^2(\rho') = S^2(\rho'_{(2)})\rho'_{(0)}S(\rho'_{(1)})$ for $\rho' \in \Gamma_L$, where $\Delta_R^2(\rho') = \rho'_{(0)} \otimes \rho'_{(1)} \otimes \rho'_{(2)}$. Observe that $\rho'_{(0)}S(\rho'_{(1)}) \in \Gamma_R$. Using (10), $g\sigma = g$ and left-covariance of g it follows that

$$\begin{aligned} g(S^2(\rho') \otimes_{\mathcal{A}} \rho) &= S^2(\rho'_{(2)})g(\rho'_{(0)}S(\rho'_{(1)}) \otimes_{\mathcal{A}} \rho) = S^2(\rho'_{(2)})g\sigma(\rho \otimes_{\mathcal{A}} \rho'_{(0)}S(\rho'_{(1)})) \\ &= S^2(\rho'_{(2)})g(\rho \otimes_{\mathcal{A}} \rho'_{(0)})S(\rho'_{(1)}) \\ &= S^2(\rho'_{(2)})S(\rho'_{(1)})g(\rho \otimes_{\mathcal{A}} \rho'_{(0)}) = g(\rho \otimes_{\mathcal{A}} \rho') \end{aligned}$$

for $\rho, \rho' \in \Gamma_L$. The other direction of the assertion can be shown similarly. \blacksquare

Now if Γ is an S^2 -invariant left-covariant FODC then we additionally require that left-covariant σ -metrics g on Γ have to satisfy (12). This in turn implies that $g(S^2(\rho)) = S^2(g(\rho))$ holds for $\rho \in \Gamma \otimes_{\mathcal{A}} \Gamma$. **In this article we always assume that the first order differential calculus (Γ, d) is self-dual.**

Remark. The property (12) of the σ -metric g will be used only in the proof of Proposition 5.3. However, the requirement $gS^2 = S^2g$ is essential for the existence of the algebra involution β of the quantum Clifford algebra $\text{Cl}(\Gamma, \sigma, g)$, see Proposition 3.2. \blacksquare

Suppose for a moment that \mathcal{A} is a coquasitriangular Hopf algebra and there is a simple corepresentation u of \mathcal{A} such that the contragredient corepresentation

u^c is isomorphic to u . For example, this is the case for $\mathcal{A} = \mathcal{O}(\mathrm{SL}_q(2))$ and u the fundamental corepresentation of \mathcal{A} . In [12] it was proved that in this setting there exists a bicovariant first order differential calculus $(\Gamma(u), d)$ over \mathcal{A} which is self-dual. Moreover, there exists a bicovariant σ -metric on $\Gamma(u)$. This example will be considered in detail in Section 6.

3 Quantum Clifford algebras

Let (Γ, d) be a FODC over \mathcal{A} , σ a braiding and g a σ -metric of Γ . Similarly to the construction in [12] one can define a contraction mapping $\langle\langle \cdot, \cdot \rangle\rangle : \Gamma^{\otimes \mathcal{A}^k} \otimes_{\mathcal{A}} \Gamma^{\otimes \mathcal{A}^l} \rightarrow \Gamma^{\otimes \mathcal{A}^{\max\{l-k, 0\}}}$. Differing from the notation therein we set $\langle\langle \rho, \rho' \rangle\rangle = 0$ for $\rho \in \Gamma^{\otimes \mathcal{A}^k}$, $\rho' \in \Gamma^{\otimes \mathcal{A}^l}$, $k > l$, and retain the formulas

$$\langle\langle \rho, \rho' \rangle\rangle = g_{12}(\rho \otimes_{\mathcal{A}} B_{1,l-1}(\rho')), \quad \langle\langle \rho \otimes_{\mathcal{A}} \rho'', \rho' \rangle\rangle = \langle\langle \rho, \langle\langle \rho'', \rho' \rangle\rangle \rangle, \quad (13)$$

where $\rho \in \Gamma$, $\rho' \in \Gamma^{\otimes \mathcal{A}^l}$, $\rho'' \in \Gamma^{\otimes \mathcal{A}}$. In [1] Bautista et. al. proposed the notion of a quantum Clifford algebra. Adapting their ideas to the present situation and having the classical situation in mind we introduce the following definition.

Definition 3.1. Let \mathcal{I} denote the two-sided ideal of the tensor algebra $\Gamma^{\otimes \mathcal{A}}$ generated by the elements

$$\{\rho_2 - g(\rho_2) \mid \rho_2 \in \ker(\mathrm{id} - \sigma)\}. \quad (14)$$

The algebra $\Gamma^{\otimes \mathcal{A}}/\mathcal{I}$ is called the *quantum Clifford algebra for the left-covariant \mathcal{A} -bimodule Γ and the σ -metric g* . We denote it by $\mathrm{Cl}(\Gamma, \sigma, g)$.

Let m_{Cl} denote the canonical mapping $m_{\mathrm{Cl}} : \Gamma^{\otimes \mathcal{A}^2}/(\mathcal{I} \cap (\mathcal{A} \oplus \Gamma^{\otimes \mathcal{A}^2}))$. Similarly to the classical case, $\mathrm{Cl}(\Gamma, \sigma, g)$ can be equipped with a \mathbb{Z}_2 -grading such that $\deg \rho_k = (-1)^k$ for $\rho_k \in \Gamma^{\otimes \mathcal{A}^k}$. Let $\mathrm{Cl}^\eta(\Gamma, \sigma, g)$, $\eta \in \{+, -\}$, denote the subspace of homogeneous elements of degree η .

Since g is left-covariant, there exists a natural left coaction of \mathcal{A} on $\mathrm{Cl}(\Gamma, \sigma, g)$.

In the classical situation, the Clifford algebra has a representation over the exterior algebra. Now we should prove that this is valid also in the quantum case.

Proposition 3.1. *The formulas*

$$a \triangleright \rho := a\rho, \quad \omega \triangleright \rho := \omega \wedge \rho + \langle\langle \omega, \rho \rangle\rangle, \quad (15)$$

where $a \in \mathcal{A}$, $\rho \in \Gamma^\wedge$ and $\omega \in \Gamma$, define a representation of $\text{Cl}(\Gamma, \sigma, g)$ over Γ^\wedge .

Proof. Since $\langle\langle \cdot, \cdot \rangle\rangle$ is a homomorphism of the left \mathcal{A} -modules $\Gamma \otimes_{\mathcal{A}} \Gamma^\wedge$ and Γ^\wedge , it is easy to see that (15) gives a well-defined action of the \mathcal{A} -bimodule Γ on Γ^\wedge . Hence this action can be extended to an action \triangleright on Γ^\wedge of the algebra $\Gamma^{\otimes \mathcal{A}}$. Now we only have to prove that the elements in (14) act trivially on Γ^\wedge .

Suppose that $a_{ij}\theta_i \otimes_{\mathcal{A}} \theta_j \in \ker(\text{id} - \sigma)$ and $\rho_k \in \Gamma^{\wedge k}$, $k \geq 0$. Then

$$\begin{aligned} (a_{ij}\theta_i \otimes_{\mathcal{A}} \theta_j - a_{ij}g_{ij}) \triangleright \rho_k &= a_{ij}\theta_i \triangleright (\theta_j \triangleright \rho_k) - a_{ij}g_{ij}\rho_k \\ &= a_{ij}\theta_i \triangleright (\theta_j \wedge \rho_k + \langle\langle \theta_j, \rho_k \rangle\rangle) - a_{ij}g_{ij}\rho_k \\ &= a_{ij}\theta_i \wedge \theta_j \wedge \rho_k + \langle\langle a_{ij}\theta_i, \theta_j \wedge \rho_k \rangle\rangle \\ &\quad + a_{ij}\theta_i \wedge \langle\langle \theta_j, \rho_k \rangle\rangle + \langle\langle a_{ij}\theta_i, \langle\langle \theta_j, \rho_k \rangle\rangle \rangle - a_{ij}g_{ij}\rho_k. \end{aligned} \quad (*)$$

Since $\langle\langle a_{ij}\theta_i, \langle\langle \theta_j, \rho_k \rangle\rangle \rangle = \langle\langle a_{ij}\theta_i \wedge \theta_j, \rho_k \rangle\rangle$, the first and fourth summands of the above expression vanish. By formula (39) in [12] the second summand can be written as

$$a_{ij}\langle\langle \theta_i, \theta_j \rangle\rangle \rho_k - b_{rs}\theta_r \wedge \langle\langle \theta_s, \rho_k \rangle\rangle, \quad (**)$$

where $b_{rs}\theta_r \otimes_{\mathcal{A}} \theta_s = \sigma^{-1}(a_{ij}\theta_i \otimes_{\mathcal{A}} \theta_j)$. The first summand of (**) is the same as the last one in (*). Therefore (*) becomes

$$-b_{rs}\theta_r \wedge \langle\langle \theta_s, \rho_k \rangle\rangle + a_{ij}\theta_i \wedge \langle\langle \theta_j, \rho_k \rangle\rangle = (a_{ij} - b_{ij})\theta_i \wedge \langle\langle \theta_j, \rho_k \rangle\rangle.$$

But $(\text{id} - \sigma^{-1})(a_{ij}\theta_i \otimes_{\mathcal{A}} \theta_j) = -\sigma^{-1}(\text{id} - \sigma)(a_{ij}\theta_i \otimes_{\mathcal{A}} \theta_j) = 0$, and so $a_{ij} = b_{ij}$ for any $i, j = 1, \dots, n$. This means that (*) vanishes for any $\rho_k \in \Gamma^{\wedge k}$. ■

Remark. Since $\rho \triangleright 1 = \rho$ for any $\rho \in \mathcal{A} \oplus \Gamma$, $\rho = 0$ in $\text{Cl}(\Gamma, \sigma, g)$ implies that $\rho = 0$ in $\mathcal{A} \oplus \Gamma$. Hence \mathcal{A} and Γ can be naturally embedded in $\text{Cl}(\Gamma, \sigma, g)$. ■

Proposition 3.2. *The involution β of the \mathcal{A} -bimodule Γ extends uniquely to an involution of the quantum Clifford algebra $\text{Cl}(\Gamma, \sigma, g)$.*

Proof. We have to show that if $(\text{id} - \sigma)\rho_2 = 0$ for $\rho_2 = a_{ij}\theta_i \otimes_{\mathcal{A}} \theta_j \in \Gamma^{\otimes \mathcal{A}^2}$ then $\beta(\theta_j)\beta(\theta_i)\beta(a_{ij}) = \beta(g(\rho_2))$ in $\text{Cl}(\Gamma, \sigma, g)$. Since $\ker(\text{id} - \sigma)$ is a left-covariant left \mathcal{A} -module, we can assume that $a_{ij} \in \mathbb{C}$. By definition we have

$$\begin{aligned} \beta(\theta_j) \otimes_{\mathcal{A}} \beta(\theta_i) &= \overline{B_j^k B_i^l} \theta_k \otimes_{\mathcal{A}} \theta_l = F_m^k \overline{E_j^m} F_n^l \overline{E_i^n} \theta_k \otimes_{\mathcal{A}} \theta_l \\ &= \overline{E_j^m E_i^n} S^2(\theta_m \otimes_{\mathcal{A}} \theta_n) = S^2(\theta_j^* \otimes_{\mathcal{A}} \theta_i^*). \end{aligned}$$

Since $S^2\sigma = \sigma S^2$ and $\sigma(\rho^*) = (\sigma^\eta(\rho))^*$ for $\rho \in \Gamma^{\otimes \mathcal{A}^2}$ with $\eta = +1$ or $\eta = -1$, it follows that $(\text{id} - \sigma)(\beta(\theta_j) \otimes_{\mathcal{A}} \beta(\theta_i) \overline{a_{ij}}) = \beta(\text{id} - \sigma^\eta)(\rho_2) = 0$. This means that $\beta(\theta_j)\beta(\theta_i)\overline{a_{ij}} = g(\beta(\theta_j) \otimes_{\mathcal{A}} \beta(\theta_i) \overline{a_{ij}})$ in $\text{Cl}(\Gamma, \sigma, g)$. But the latter is equal to $\beta(g(\rho_2))$ because of $gS^2 = S^2g$ and the requirement (11). \blacksquare

Similarly to the classical situation a left-covariant left ideal \mathcal{S} of $\text{Cl}(\Gamma, \sigma, g)$ is called a *spinor module* if \mathcal{S} is an \mathcal{A} -bimodule, too. By Theorem 4.1.1. in [20] it follows that \mathcal{S} is a free left \mathcal{A} -module and there exists a left-invariant \mathcal{A} -module basis of \mathcal{S} . Let $m_{\text{Cl}, \mathcal{S}}$ denote the left action $m_{\text{Cl}, \mathcal{S}} : \Gamma \otimes_{\mathcal{A}} \mathcal{S} \rightarrow \mathcal{S}$ of Γ on the spinor module \mathcal{S} .

One of the most important aims of this article is to introduce a (hermitean non-degenerate) metric on spinor modules \mathcal{S} . In order to do this let \mathcal{S}^c denote the complex conjugate of the vector space \mathcal{S} , that is $\mathcal{S}^c = \mathcal{S}$ as sets and $\lambda\psi^c = (\bar{\lambda}\psi)^c$ for $\lambda \in \mathbb{C}$, $\psi \in \mathcal{S}$. We define a left coaction Δ_L and an \mathcal{A} -bimodule structure on the vector space $\mathcal{S} \otimes \mathcal{S}^c$ by

$$\begin{aligned} \Delta_L(\psi \otimes \psi'^c) &:= \psi_{(-1)}(\psi'_{(-1)})^* \otimes (\psi_{(0)} \otimes \psi'_{(0)}{}^c), \\ a(\psi \otimes \psi'^c) &= a\psi \otimes \psi'^c, \quad (\psi \otimes \psi'^c)a = \psi \otimes a^*\psi'^c \end{aligned} \tag{16}$$

for $\psi, \psi' \in \mathcal{S}$, $a \in \mathcal{A}$, where $\Delta_L(\psi) = \psi_{(-1)} \otimes \psi_{(0)}$, $\Delta_L(\psi') = \psi'_{(-1)} \otimes \psi'_{(0)}$ and $*$ is the involution of \mathcal{A} .

Definition 3.2. A mapping $\langle \cdot, \cdot \rangle : \mathcal{S} \otimes \mathcal{S}^c \rightarrow \mathbb{C}$ is called a *metric on \mathcal{S}* if

(i) there exists a left-covariant mapping $\langle \cdot, \cdot \rangle_0 : \mathcal{S} \otimes \mathcal{S}^c \rightarrow \mathcal{A}$ such that both $\langle a\psi, (b\psi')^c \rangle_0 = a\langle \psi, \psi' \rangle_0 b^*$ for any $a, b \in \mathcal{A}$ and $\psi, \psi' \in \mathcal{S}$ and the equation $\langle \cdot, \cdot \rangle = h \circ \langle \cdot, \cdot \rangle_0$ hold,

(ii) $\langle \cdot, \cdot \rangle$ is non-degenerate and hermitean (but it needs not be positive definite).

To simplify the notation we will write $\langle a\psi, b\psi' \rangle$ for $\langle a\psi, (b\psi')^c \rangle$. But remember that the metric is antilinear in the second component.

Remarks. 1. Let $\langle \cdot, \cdot \rangle_0 : \mathcal{S} \otimes \mathcal{S}^c \rightarrow \mathcal{A}$ be such that $\langle a\psi, b\psi' \rangle_0 = a\langle \psi, \psi' \rangle_0 b^*$ for any $a, b \in \mathcal{A}$ and $\psi, \psi' \in \mathcal{S}$. Then it is left-covariant if and only if $\langle \psi, \psi' \rangle_0 \in \mathbb{C}$ for any $\psi, \psi' \in \mathcal{S}_L$. Hence, if $\langle \cdot, \cdot \rangle$ is a metric on \mathcal{S} then $\langle \cdot, \cdot \rangle_0$ can be reconstructed by the setting $\langle a\psi, b\psi' \rangle_0 := a\langle \psi, \psi' \rangle_0 b^*$ for $a, b \in \mathcal{A}$, $\psi, \psi' \in \mathcal{S}_L$.

2. A mapping $\langle \cdot, \cdot \rangle$ in (i) of Definition 3.2 is non-degenerate if and only if its restriction $\langle \cdot, \cdot \rangle : \mathcal{S}_L \otimes \mathcal{S}_L^c \rightarrow \mathbb{C}$ is non-degenerate and

3. it is hermitean if and only if its restriction $\langle \cdot, \cdot \rangle : \mathcal{S}_L \otimes \mathcal{S}_L^c \rightarrow \mathbb{C}$ is hermitean.

4. Any metric $\langle \cdot, \cdot \rangle$ on \mathcal{S} satisfies the equations

$$\langle a\psi, b\psi' \rangle_0^* = \langle b\psi', a\psi \rangle_0 \quad \text{and} \quad (17)$$

$$\langle a\psi, \psi' \rangle = \langle \psi, \beta(a)\psi' \rangle \quad (18)$$

for all $a \in \mathcal{A}$ and $\psi, \psi' \in \mathcal{S}$. ■

Of course, metrics on spinor modules always exist. In this paper interesting metrics will also have to satisfy the equation

$$\langle u\varphi, \psi \rangle = \langle \varphi, \beta(u)\psi \rangle \quad (19)$$

for $u \in \text{Cl}(\Gamma, \sigma, g)$ and $\varphi, \psi \in \mathcal{S}$ (see Proposition 5.2). A similar construction as in Section 8.2.1 in [5] gives the existence of such a metric on "minimal" spinor modules (minimal left ideals).

Observe that $\beta(\mathcal{S})$ is a minimal right ideal. If $\text{Cl}(\Gamma, \sigma, g)$ is a simple left-covariant algebra, then $\beta(\mathcal{S}_L)\mathcal{S}_L = \beta(\mathcal{S}_L) \cap \mathcal{S}_L$ is a one-dimensional subspace of \mathcal{S}_L . Since $\beta^2 = \text{id}$, it contains an element x such that $\beta(x) = x$. Consider elements $\psi, \psi' \in \mathcal{S}$. Since \mathcal{S} is an \mathcal{A} -bimodule, there exist $a, b \in \mathcal{A}$ such that $\beta(\psi')\psi = ax = xb$. Hence $xc = c_{(1)}\hat{f}(c_{(2)})x$ for all $c \in \mathcal{A}$ and therefore $h(a) = h(b)$. We define $\langle \psi, \psi' \rangle = h(a) = h(b)$. Since

$$\beta(\psi)\psi' = \beta(\beta(\psi')\psi) = \beta(ax) = x\beta(a),$$

we obtain $\langle \psi', \psi \rangle = h(\beta(a)) = h(\rho(a)^*) = \overline{h(a)}$. This means that the mapping $\langle \cdot, \cdot \rangle$ is hermitean. For an arbitrary element $u \in \text{Cl}(\Gamma, \sigma, g)$ we get

$$\beta(\psi')u\psi = \beta(\beta(u)\psi')\psi$$

and therefore $\langle u\psi, \psi' \rangle = \langle \psi, \beta(u)\psi' \rangle$. Finally we have to prove the non-degeneracy of $\langle \cdot, \cdot \rangle$. Since $\beta(\mathcal{S}_L)\mathcal{S}_L$ is a nontrivial vector space, there are $\psi', \psi'' \in \mathcal{S}_L$ such that $\beta(\psi'')\psi' \neq 0$. Because \mathcal{S} is a minimal left ideal of $\text{Cl}(\Gamma, \sigma, g)$, for any $\psi \in \mathcal{S}_L$, $\psi \neq 0$, there exists $u \in \text{Cl}(\Gamma, \sigma, g)$ such that $\psi'' = u\psi$. Then $0 \neq \langle \psi', \psi'' \rangle = \langle \psi', u\psi \rangle = \langle \beta(u)\psi', \psi \rangle$ and hence $\langle \cdot, \cdot \rangle$ is non-degenerate on $\mathcal{S}_L \otimes \mathcal{S}_L^c$. This means that there are bases $\{\psi_i\}$ and $\{\psi'_i\}$ of \mathcal{S}_L such that $\langle \psi_i, \psi'_j \rangle = \delta_{ij}$.

Suppose now that $\psi \in \mathcal{S} \setminus \{0\}$. Then $\psi = \psi_i a_i$, $a_i \in \mathcal{A}$, and we can assume that $a := a_1 \neq 0$. There exists $b \in \mathcal{A}$ such that $h(ba_{(1)}\hat{f}(a_{(2)})) \neq 0$. We conclude that

$$\beta(\psi'_1\beta(b))\psi = b\beta(\psi'_1)\psi_i a_i = bxa = ba_{(1)}\hat{f}(a_{(2)})x$$

from which $\langle \psi, \psi'_1 \beta(b) \rangle \neq 0$ follows. This proves the following proposition.

Proposition 3.3. *Suppose that $\text{Cl}(\Gamma, \sigma, g)$ is a simple left-covariant algebra and \mathcal{S} is a minimal left ideal of $\text{Cl}(\Gamma, \sigma, g)$ and an \mathcal{A} -bimodule. Then there exists a metric on \mathcal{S} such that $\langle u\psi, \psi' \rangle = \langle \psi, \beta(u)\psi' \rangle$ holds for $u \in \text{Cl}(\Gamma, \sigma, g)$, $\psi, \psi' \in \mathcal{S}$.*

4 Connections

Let \mathcal{C} be an \mathcal{A} -bimodule and (Γ, d) a first order differential calculus over \mathcal{A} . Following [2] we call a map $\nabla : \mathcal{C} \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{C}$ a *left connection on \mathcal{C}* if it satisfies the rule

$$\nabla(a\rho) = da \otimes_{\mathcal{A}} \rho + a \nabla(\rho) \quad (20)$$

for any $a \in \mathcal{A}$ and any $\rho \in \mathcal{C}$. If \mathcal{C} is a left-covariant \mathcal{A} -bimodule then a left connection ∇ on \mathcal{C} is called *left-covariant*, if $\Delta_L(\nabla(\rho)) = (\text{id} \otimes \nabla)\Delta_L(\rho)$.

Unfortunately, in general it is not possible to extend a connection on an \mathcal{A} -bimodule \mathcal{C} to the tensor product $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}$ ($\mathcal{C}^{\otimes_{\mathcal{A}}}$, respectively). In [2] a constraint was given in which case such an extension can be made. The definition therein yields that a left connection is extensible if and only if there exists a bimodule homomorphism $\tilde{\sigma} : \mathcal{C} \otimes_{\mathcal{A}} \Gamma \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{C}$ such that

$$\tilde{\sigma}(\rho \otimes_{\mathcal{A}} da) = \nabla(\rho a) - \nabla(\rho)a \quad (21)$$

holds for all $a \in \mathcal{A}$ and $\rho \in \mathcal{C}$. In this case, following Mourad's definition [19, 9], ∇ is called a *linear left connection on the \mathcal{A} -bimodule \mathcal{C}* .

Let ∇ be a linear left connection on \mathcal{C} . One defines the left connection $\nabla : \mathcal{C}^{\otimes_{\mathcal{A}} k} \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{C}^{\otimes_{\mathcal{A}} k}$, $k \geq 2$, recursively by

$$\nabla(\rho \otimes_{\mathcal{A}} \rho_{k-1}) := \nabla(\rho) \otimes_{\mathcal{A}} \rho_{k-1} + (\tilde{\sigma} \otimes_{\mathcal{A}} \text{id}^{k-1})(\rho \otimes_{\mathcal{A}} \nabla(\rho_{k-1})) \quad (22)$$

for $\rho \in \mathcal{C}$ and $\rho_{k-1} \in \mathcal{C}^{\otimes_{\mathcal{A}} k-1}$. Moreover, if \mathcal{C}' is another \mathcal{A} -bimodule and ∇' is a left connection on \mathcal{C}' then the formula

$$\tilde{\nabla}(\rho \otimes_{\mathcal{A}} \rho') := \nabla(\rho) \otimes_{\mathcal{A}} \rho' + (\tilde{\sigma} \otimes_{\mathcal{A}} \text{id})(\rho \otimes_{\mathcal{A}} \nabla'(\rho')), \quad (23)$$

$\rho \in \mathcal{C}$, $\rho' \in \mathcal{C}'$, defines a left connection on $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}'$. If additionally ∇' is an extensible left connection on \mathcal{C}' , $\nabla'(\rho'a) = \nabla'(\rho')a + \tilde{\sigma}'(\rho' \otimes_{\mathcal{A}} da)$, then $\tilde{\nabla}$ satisfies the equation

$$\tilde{\nabla}(\rho \otimes_{\mathcal{A}} \rho' a) = \tilde{\nabla}(\rho \otimes_{\mathcal{A}} \rho')a + (\tilde{\sigma} \otimes \text{id})(\text{id} \otimes \tilde{\sigma}')(\rho \otimes_{\mathcal{A}} \rho' \otimes_{\mathcal{A}} da) \quad (24)$$

for $\rho \in \mathcal{C}, \rho' \in \mathcal{C}'$ and $a \in \mathcal{A}$.

Let \mathcal{C} be a left-covariant \mathcal{A} -bimodule and let $\{\eta_\alpha \mid \alpha = 1, \dots, M := \dim \mathcal{C}\}$ be a basis of $\mathcal{C}_L := \{\eta \in \mathcal{C} \mid \Delta_L(\eta) = 1 \otimes \eta\}$. There are linear functionals F_β^α on \mathcal{A} such that $\eta_\alpha a = a_{(1)} F_\beta^\alpha(a_{(2)}) \eta_\beta$ for any $a \in \mathcal{A}, \alpha = 1, \dots, M$.

Lemma 4.1. *Let ∇ be a left-covariant left connection on \mathcal{C} , $\nabla(\eta_\alpha) = \Gamma_\alpha^{i\beta} \theta_i \otimes_{\mathcal{A}} \eta_\beta$, $\Gamma_\alpha^{i\beta} \in \mathbb{C}$. Then ∇ is a linear left connection on \mathcal{C} if and only if*

$$\Gamma_\alpha^{i\beta} \varepsilon + S(F_\gamma^\alpha) X_i F_\beta^\gamma - S(F_\delta^\alpha) \Gamma_\delta^{j\gamma} f_i^j F_\beta^\gamma \in \mathcal{X} \quad (25)$$

for any $\alpha, \beta = 1, \dots, M, i = 1, \dots, n$.

Proof. Recall that $\{\theta_i \otimes_{\mathcal{A}} \eta_\beta\}$ and $\{\eta_\alpha\}$ are free bases of the left \mathcal{A} -modules $\Gamma \otimes_{\mathcal{A}} \mathcal{C}$ and \mathcal{C} , respectively. Hence by (20) any left connection on \mathcal{C} is uniquely determined by elements $\Gamma_\alpha^{i\beta} \in \mathcal{A}$. Moreover, left-covariance of ∇ is satisfied if and only if $\Gamma_\alpha^{i\beta} \in \mathbb{C}$ for any $\alpha, \beta = 1, \dots, \dim \mathcal{C}$ and $i = 1, \dots, n$.

By the above considerations and because of $da = a_{(1)} S(a_{(2)}) da_{(3)}$, ∇ is a linear left connection if and only if the mapping $\tilde{\sigma}$, given by

$$\begin{aligned} \tilde{\sigma}(\rho_i S(a_{i(1)}) \otimes_{\mathcal{A}} da_{i(2)}) &:= \nabla(\rho_i S(a_{i(1)}) a_{i(2)}) - \nabla(\rho_i S(a_{i(1)})) a_{i(2)} \\ &= \varepsilon(a_i) \nabla(\rho_i) - \nabla(\rho_i S(a_{i(1)})) a_{i(2)}, \end{aligned} \quad (26)$$

$\rho_i \in \mathcal{C}, a_i \in \mathcal{A}$, is well defined. On the left hand side the expression $\tilde{\sigma}(\rho_i \otimes_{\mathcal{A}} S(a_{i(1)}) da_{i(2)}) = \tilde{\sigma}(\rho_i \otimes_{\mathcal{A}} \omega(a_i))$ appears which is zero for linearly independent ρ_i if and only if $a_i \in \mathcal{R}_\Gamma + \mathbb{C} \cdot 1$. This gives that $\tilde{\sigma}$ is well-defined if and only if $\varepsilon(a) \nabla(\rho) - \nabla(\rho S(a_{(1)})) a_{(2)} = 0$ for any $a \in \mathcal{R}_\Gamma + \mathbb{C} \cdot 1$ and $\rho \in \mathcal{C}_L$. Now let us compute this expression for $\rho = \eta_\alpha$.

$$\begin{aligned} \varepsilon(a) \nabla(\eta_\alpha) - \nabla(\eta_\alpha S(a_{(1)})) a_{(2)} &= \\ &= \varepsilon(a) \Gamma_\alpha^{i\beta} \theta_i \otimes_{\mathcal{A}} \eta_\beta - \nabla(S(a_{(2)}) F_\gamma^\alpha(S(a_{(1)})) \eta_\gamma) a_{(3)} \\ &= \varepsilon(a) \Gamma_\alpha^{i\beta} \theta_i \otimes_{\mathcal{A}} \eta_\beta - dS(a_{(2)}) S(F_\gamma^\alpha)(a_{(1)}) \otimes_{\mathcal{A}} \eta_\gamma a_{(3)} \\ &\quad - S(a_{(2)}) S(F_\delta^\alpha)(a_{(1)}) \Gamma_\delta^{j\gamma} \theta_j \otimes_{\mathcal{A}} \eta_\gamma a_{(3)}. \end{aligned} \quad (*)$$

The second summand of $(*)$ can be reformulated as

$$\begin{aligned} -dS(a_{(2)}) S(F_\gamma^\alpha)(a_{(1)}) \otimes_{\mathcal{A}} a_{(3)} F_\beta^\gamma(a_{(4)}) \eta_\beta &= \\ &= (-d(S(a_{(2)}) a_{(3)}) + S(a_{(2)}) da_{(3)}) S(F_\gamma^\alpha)(a_{(1)}) F_\beta^\gamma(a_{(4)}) \otimes_{\mathcal{A}} \eta_\beta \\ &= S(a_{(2)}) a_{(3)} X_i(a_{(4)}) \theta_i S(F_\gamma^\alpha)(a_{(1)}) F_\beta^\gamma(a_{(5)}) \otimes_{\mathcal{A}} \eta_\beta \\ &= (S(F_\gamma^\alpha) X_i F_\beta^\gamma)(a) \theta_i \otimes_{\mathcal{A}} \eta_\beta. \end{aligned}$$

For the third summand of $(*)$ one computes

$$\begin{aligned} & -\Gamma_\delta^{j\gamma} S(a_{(2)}) S(F_\delta^\alpha)(a_{(1)}) a_{(3)} (f_i^j F_\beta^\gamma)(a_{(4)}) \theta_i \otimes_{\mathcal{A}} \eta_\beta = \\ & = -\Gamma_\delta^{j\gamma} (S(F_\delta^\alpha) f_i^j F_\beta^\gamma)(a) \theta_i \otimes_{\mathcal{A}} \eta_\beta. \end{aligned}$$

Together we obtain

$$\begin{aligned} \varepsilon(a) \nabla(\eta_\alpha) - \nabla(\eta_\alpha S(a_{(1)})) a_{(2)} = \\ = (\Gamma_\alpha^{i\beta} \varepsilon + S(F_\gamma^\alpha) X_i F_\beta^\gamma - S(F_\delta^\alpha) \Gamma_\delta^{j\gamma} f_i^j F_\beta^\gamma)(a) \theta_i \otimes_{\mathcal{A}} \eta_\beta \end{aligned} \quad (27)$$

for any $a \in \mathcal{A}$ and $\alpha = 1, \dots, M$. Finally, recall that $f \in \mathcal{A}'$, $f(a) = 0$ for any $a \in \mathcal{R}_\Gamma + \mathbb{C} \cdot 1$ is by definition equivalent to $f \in \mathcal{X}$. \blacksquare

Definition 4.1. Let \mathcal{C} be an \mathcal{A} -bimodule algebra with multiplication \circ . Then a linear mapping $\nabla : \mathcal{C} \rightarrow \Gamma \otimes \mathcal{C}$ is called a *linear left connection on \mathcal{C}* , if

- (i) ∇ is a left connection on the \mathcal{A} -bimodule \mathcal{C} ,
- (ii) there exists a bimodule homomorphism $\tilde{\sigma} : \mathcal{C} \otimes \Gamma \rightarrow \Gamma \otimes \mathcal{C}$ such that $\tilde{\sigma}(\rho \otimes_{\mathcal{A}} da) = \nabla(\rho a) - \nabla(\rho) a$ for any $a \in \mathcal{A}$, $\rho \in \mathcal{C}$, and
- (iii) $\nabla(\rho \circ \rho') = (\text{id} \otimes_{\mathcal{A}} \circ)(\nabla(\rho) \otimes_{\mathcal{A}} \rho' + (\tilde{\sigma} \otimes_{\mathcal{A}} \text{id})(\rho \otimes_{\mathcal{A}} \nabla(\rho')))$ holds for any $\rho, \rho' \in \mathcal{C}$.

Let \mathcal{C} be an \mathcal{A} -bimodule algebra and \mathcal{S} a left \mathcal{C} -module with left action \triangleright . Then a mapping $\nabla_{\mathcal{S}} : \mathcal{S} \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{S}$ is called a *linear left connection on \mathcal{S}* if there is a left connection ∇ on \mathcal{C} such that

$$\nabla_{\mathcal{S}}(a\varphi) = (\text{id} \otimes_{\mathcal{A}} \triangleright)(\nabla(a) \otimes_{\mathcal{A}} \varphi + (\tilde{\sigma} \otimes_{\mathcal{A}} \text{id})(a \otimes_{\mathcal{A}} \nabla_{\mathcal{S}}(\varphi))) \quad (28)$$

for any $a \in \mathcal{C}$ and $\varphi \in \mathcal{S}$.

Let us call a FODC (Γ, d) over \mathcal{A} *inner*, if there exists an $\omega \in \Gamma$ such that $da = \omega a - a\omega$ for any $a \in \mathcal{A}$.

Lemma 4.2. Let \mathcal{C} be an \mathcal{A} -bimodule and let (Γ, d) be an inner FODC over \mathcal{A} such that $da = \omega a - a\omega$ for $a \in \mathcal{A}$. Let $\tilde{\sigma} : \mathcal{C} \otimes_{\mathcal{A}} \Gamma \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{C}$ and $V : \mathcal{C} \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{C}$ be homomorphisms of \mathcal{A} -bimodules. Then the assignment

$$\nabla(\rho) := \omega \otimes_{\mathcal{A}} \rho - \tilde{\sigma}(\rho \otimes_{\mathcal{A}} \omega) + V(\rho) \quad (29)$$

defines a linear left connection ∇ on \mathcal{C} such that $\nabla(\rho a) - \nabla(\rho) a = \tilde{\sigma}(\rho \otimes_{\mathcal{A}} da)$. Moreover, any linear left connection on \mathcal{C} is given in this manner.

Remark. For $\mathcal{C} = \Gamma$ the assertion of the lemma was already proved in [2, Prop. C.3]. Since the proof of this lemma is similar, we omit it. \blacksquare

For inner first order differential calculi Γ over \mathcal{A} the following proposition holds.

Proposition 4.3. *Suppose that (Γ, d) is an inner FODC and ∇ is a linear left connection on Γ such that $\nabla(\rho) = \omega \otimes_{\mathcal{A}} \rho - \tilde{\sigma}(\rho \otimes_{\mathcal{A}} \omega)$, $\rho \in \Gamma$, for an endomorphism $\tilde{\sigma}$ of the \mathcal{A} -bimodule $\Gamma \otimes_{\mathcal{A}} \Gamma$.*

1. *The linear left connection $\nabla : \Gamma^{\otimes_{\mathcal{A}}} \rightarrow \Gamma \otimes_{\mathcal{A}} \Gamma^{\otimes_{\mathcal{A}}}$ is compatible with Woronowicz' antisymmetrizer if $\sigma_{23} \tilde{\sigma}_{12} \tilde{\sigma}_{23} = \tilde{\sigma}_{12} \tilde{\sigma}_{23} \sigma_{12}$.*
2. *The linear left connection $\nabla : \Gamma^{\otimes_{\mathcal{A}}} \rightarrow \Gamma \otimes_{\mathcal{A}} \Gamma^{\otimes_{\mathcal{A}}}$ is compatible with the multiplication of the Clifford algebra if $\sigma_{23} \tilde{\sigma}_{12} \tilde{\sigma}_{23} = \tilde{\sigma}_{12} \tilde{\sigma}_{23} \sigma_{12}$ and $g_{23} \tilde{\sigma}_{12} \tilde{\sigma}_{23} = g_{12}$.*

Proof. First one checks with (22) that

$$\nabla(\rho_k) = \omega \otimes_{\mathcal{A}} \rho_k - \tilde{\sigma}_{12} \tilde{\sigma}_{23} \cdots \tilde{\sigma}_{k,k+1}(\rho_k \otimes_{\mathcal{A}} \omega) \quad (30)$$

for any $\rho_k \in \Gamma^{\otimes_{\mathcal{A}^k}}$. Let $\tilde{\sigma} : \Gamma^{\otimes_{\mathcal{A}}} \otimes_{\mathcal{A}} \Gamma \rightarrow \Gamma \otimes_{\mathcal{A}} \Gamma^{\otimes_{\mathcal{A}}}$ denote the linear mapping defined by

$$\tilde{\sigma}(\rho_k \otimes_{\mathcal{A}} \rho) := \tilde{\sigma}_{12} \tilde{\sigma}_{23} \cdots \tilde{\sigma}_{k,k+1}(\rho_k \otimes_{\mathcal{A}} \rho), \quad \rho_k \in \Gamma^{\otimes_{\mathcal{A}^k}}, \rho \in \Gamma. \quad (31)$$

Suppose that $\sigma_{23} \tilde{\sigma}_{12} \tilde{\sigma}_{23} = \tilde{\sigma}_{12} \tilde{\sigma}_{23} \sigma_{12}$. Then for any $i \in \mathbb{N}$, $1 < i \leq k$ we have $\sigma_{i,i+1} \tilde{\sigma}_{12} \tilde{\sigma}_{23} \cdots \tilde{\sigma}_{k,k+1} = \tilde{\sigma}_{12} \tilde{\sigma}_{23} \cdots \tilde{\sigma}_{k,k+1} \sigma_{i-1,i}$ and hence

$$(\text{id} \otimes_{\mathcal{A}} A_k) \tilde{\sigma}_{12} \tilde{\sigma}_{23} \cdots \tilde{\sigma}_{k,k+1} = \tilde{\sigma}_{12} \tilde{\sigma}_{23} \cdots \tilde{\sigma}_{k,k+1} (A_k \otimes_{\mathcal{A}} \text{id}) \quad \text{for any } k \geq 2. \quad (32)$$

Similarly, if $g_{23} \tilde{\sigma}_{12} \tilde{\sigma}_{23} = g_{12}$ and $k \geq 2$ then for any $i \in \mathbb{N}$, $1 < i \leq k$ we have $g_{i,i+1} \tilde{\sigma}_{12} \cdots \tilde{\sigma}_{k,k+1} = \tilde{\sigma}_{12} \cdots \tilde{\sigma}_{k-2,k-1} g_{i-1,i}$.

Suppose now that $A_k(\rho_k) = 0$ for a $\rho_k \in \Gamma^{\otimes_{\mathcal{A}^k}}$, $k \geq 2$. Then

$$\begin{aligned} (\text{id} \otimes_{\mathcal{A}} A_k) \nabla(\rho_k) &= \omega \otimes_{\mathcal{A}} A_k(\rho_k) - (\text{id} \otimes_{\mathcal{A}} A_k) \tilde{\sigma}(\rho_k \otimes_{\mathcal{A}} \omega) \\ &= - \tilde{\sigma}(A_k(\rho_k) \otimes_{\mathcal{A}} \omega) = 0. \end{aligned}$$

Hence ∇ is a well-defined connection on Γ^\wedge .

Let now $\rho \in \Gamma^{\otimes_{\mathcal{A}}}$ be an element of the ideal \mathcal{I} in Definition 3.1. Without loss of generality we may assume that $\rho = \rho' \otimes_{\mathcal{A}} (\rho_2 - g(\rho_2)) \otimes_{\mathcal{A}} \rho''$, where $\rho' \in \Gamma^{\otimes_{\mathcal{A}^k}}$, $\rho'' \in \Gamma^{\otimes_{\mathcal{A}^l}}$, $k, l \in \mathbb{N}_0$ and $\rho_2 \in \Gamma^{\otimes_{\mathcal{A}^2}}$, $\sigma(\rho_2) = \rho_2$. Then $\nabla(\rho) = \omega \otimes_{\mathcal{A}} \rho - \tilde{\sigma}(\rho \otimes_{\mathcal{A}} \omega)$

and the first summand is an element of $\Gamma \otimes_{\mathcal{A}} \mathcal{I}$. Now it suffices to show that

$$(\text{id} - \sigma_{k+2,k+3})\tilde{\sigma}((\rho' \otimes_{\mathcal{A}} \rho_2 \otimes_{\mathcal{A}} \rho'') \otimes_{\mathcal{A}} \omega) = 0 \quad \text{and} \quad (33)$$

$$g_{k+2,k+3}\tilde{\sigma}((\rho' \otimes_{\mathcal{A}} \rho_2 \otimes_{\mathcal{A}} \rho'') \otimes_{\mathcal{A}} \omega) = \tilde{\sigma}((\rho' g(\rho_2) \otimes_{\mathcal{A}} \rho'') \otimes_{\mathcal{A}} \omega). \quad (34)$$

But (33) follows from $\sigma_{k+2,k+3}\tilde{\sigma} = \tilde{\sigma}\sigma_{k+1,k+2}$ and $\sigma\rho_2 = \rho_2$ and (34) is proved by $g_{k+2,k+3}\tilde{\sigma} = \tilde{\sigma}g_{k+1,k+2}$. \blacksquare

Proposition 4.4. *Let (Γ, d) be an inner FODC over \mathcal{A} , $da = \omega a - a\omega$ for $a \in \mathcal{A}$. Let \mathcal{C} be an \mathcal{A} -bimodule and left $\Gamma^{\otimes \mathcal{A}}$ -module with left action m . Suppose that ∇ and ∇' are linear left connections on $\Gamma^{\otimes \mathcal{A}}$ and on the \mathcal{A} -bimodule \mathcal{C} , respectively, such that*

$$\nabla(\rho) = \omega \otimes_{\mathcal{A}} \rho - \tilde{\sigma}(\rho \otimes_{\mathcal{A}} \omega), \quad \nabla'(\psi) = \omega \otimes_{\mathcal{A}} \psi - \tau(\psi \otimes_{\mathcal{A}} \omega) + V(\psi) \quad (35)$$

holds for $\rho \in \Gamma^{\otimes \mathcal{A}}, \psi \in \mathcal{C}$. Then ∇' is compatible with the left action m of $\Gamma^{\otimes \mathcal{A}}$ on \mathcal{C} if and only if

$$(\tau m_{12} - m_{23}\tilde{\sigma}_{12}\tau_{23})(\rho \otimes_{\mathcal{A}} \psi \otimes_{\mathcal{A}} \omega) + (m_{23}\tilde{\sigma}_{12}V_2 - Vm)(\rho \otimes_{\mathcal{A}} \psi) = 0 \quad (36)$$

for all $\rho \in \Gamma, \psi \in \mathcal{C}$.

Proof. It is enough to compare both sides of (28) for $a \in \Gamma$ and $\varphi \in \mathcal{C}$. The left hand side becomes $\omega \otimes_{\mathcal{A}} a\varphi - \tau(a\varphi \otimes_{\mathcal{A}} \omega) + V(a\varphi)$. The right hand side takes the form

$$\begin{aligned} m_{23}(\nabla(a) \otimes_{\mathcal{A}} \varphi + \tilde{\sigma}_{12}(a \otimes_{\mathcal{A}} \nabla' \varphi)) &= \\ &= m_{23}(\omega \otimes_{\mathcal{A}} a \otimes_{\mathcal{A}} \varphi - \tilde{\sigma}_{12}(a \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} \varphi) \\ &\quad + \tilde{\sigma}_{12}(a \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} \varphi) - \tilde{\sigma}_{12}\tau_{23}(a \otimes_{\mathcal{A}} \varphi \otimes_{\mathcal{A}} \omega) + \tilde{\sigma}_{12}V_2(a \otimes_{\mathcal{A}} \varphi)) \\ &= \omega \otimes_{\mathcal{A}} a\varphi - m_{23}\tilde{\sigma}_{12}\tau_{23}(a \otimes_{\mathcal{A}} \varphi \otimes_{\mathcal{A}} \omega) + m_{23}\tilde{\sigma}_{12}V_2(a \otimes_{\mathcal{A}} \varphi). \end{aligned}$$

Both sides are equal if and only if (36) holds. \blacksquare

Let Γ^{\wedge} be a differential calculus over \mathcal{A} with first order part Γ . Let ∇ be a left connection on Γ . The mapping $T : \Gamma \rightarrow \Gamma^{\wedge 2}$, defined by the formula $T := m_{\wedge}\nabla - d$, is called the *torsion* of ∇ . It satisfies $T(a\rho) = aT(\rho)$ for all $a \in \mathcal{A}, \rho \in \Gamma$. If ∇ is a linear left connection on Γ , $\nabla(\rho a) = \nabla(\rho)a + \tilde{\sigma}(\rho \otimes_{\mathcal{A}} da)$, then T is a homomorphism of \mathcal{A} -bimodules if and only if

$$m_{\wedge}(\text{id} + \tilde{\sigma})(\rho \otimes_{\mathcal{A}} \rho') = 0 \quad (37)$$

for all $\rho, \rho' \in \Gamma$.

Suppose that Γ^\wedge is an inner differential calculus over \mathcal{A} , that is there exists $\omega \in \Gamma$ such that $d\rho = \omega \wedge \rho - (-1)^k \rho \wedge \omega$ for all $\rho \in \Gamma^{\wedge k}$. Let ∇ be a linear left connection on Γ which satisfies $\nabla \rho = \omega \otimes_{\mathcal{A}} \rho - \tilde{\sigma}(\rho \otimes_{\mathcal{A}} \omega)$ for all $\rho \in \Gamma$. If the torsion T of ∇ fulfills (37) then $T(\rho) = 0$ for all $\rho \in \Gamma$.

Let \mathcal{S} be a spinor module, $\langle \cdot, \cdot \rangle$ a metric on \mathcal{S} and let ∇ and $\nabla_{\mathcal{S}}$ be linear left connections on $\text{Cl}(\Gamma, \sigma, g)$ and on \mathcal{S} , respectively. Generalizing the notion of Definition 3.2, we use the symbol $\langle a\psi, b\psi' \rangle_0 := a \otimes_{\mathcal{A}} \langle \psi, \psi' \rangle_0 b^*$ for all $a, b \in \text{Cl}(\Gamma, \sigma, g)$, $\psi, \psi' \in \mathcal{S}$. Then the mapping $\nabla_{\mathcal{S}}^* : \mathcal{S} \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{S}$, defined by

$$\langle \psi, \nabla_{\mathcal{S}}^*(\psi') \rangle_0 := d\langle \psi, \psi' \rangle_0 - \langle \nabla_{\mathcal{S}} \psi, \psi' \rangle_0, \quad \psi, \psi' \in \mathcal{S}, \quad (38)$$

is a left connection on \mathcal{S} . It is called the *connection dual to $\nabla_{\mathcal{S}}$* (see also [2]). The connection dual to $\nabla_{\mathcal{S}}^*$, denoted by $\nabla_{\mathcal{S}}^{**}$, is again $\nabla_{\mathcal{S}}$ itself. Indeed, applying the involution $*$ onto (38) and having Remark 1 after Definition 3.2 in mind we obtain

$$\langle \nabla_{\mathcal{S}}^*(\psi'), \psi \rangle_0 = d\langle \psi', \psi \rangle_0 - \langle \psi', \nabla_{\mathcal{S}} \psi \rangle_0, \quad \psi, \psi' \in \mathcal{S}. \quad (39)$$

Hence $\langle \psi', \nabla_{\mathcal{S}} \psi \rangle_0 = \langle \psi', \nabla_{\mathcal{S}}^{**} \psi \rangle_0$ for all $\psi, \psi' \in \mathcal{S}$.

Remark. It is not clear whether the connection dual to $\nabla_{\mathcal{S}}$ is linear or/and compatible with the left multiplication of the quantum Clifford algebra. The examples in Section 6 show that this can be the case, but $\nabla_{\mathcal{S}}$ and $\nabla_{\mathcal{S}}^*$ are not necessarily compatible with the same linear connection on $\text{Cl}(\Gamma, \sigma, g)$. ■

The mapping $\nabla^* \nabla_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{S}$, given by

$$\langle \nabla^* \nabla_{\mathcal{S}}(\psi), \psi' \rangle := -hg(\langle \nabla_{\mathcal{S}} \psi, \nabla_{\mathcal{S}} \psi' \rangle_0), \quad \psi, \psi' \in \mathcal{S}, \quad (40)$$

is called the *connection Laplacian* associated to the connection $\nabla_{\mathcal{S}}$ on \mathcal{S} .

5 Dirac operators

Let \mathcal{C} be a finite dimensional left-covariant \mathcal{A} -bimodule with basis $\{\eta_i \mid i = 1, \dots, p\}$ of \mathcal{C}_L . Let $\mathcal{X} = \text{Lin}\{X_j \mid j = 1, \dots, n\}$ and $\langle \mathcal{X} \rangle$ denote the tangent space of a left-covariant FODC (Γ, d) over \mathcal{A} and the unital complex subalgebra of \mathcal{A}° generated by \mathcal{X} , respectively. Observe that $\langle \mathcal{X} \rangle$ can be equipped with a filtration such that $\deg X_j = 1$ for all $j = 1, \dots, n$.

Definition 5.1. A mapping $\partial : \mathcal{C} \rightarrow \mathcal{C}$ is called a *left-invariant differential operator* on \mathcal{C} (with respect to the differential calculus (Γ, d)) if there exist functionals $p_{i,j} \in \langle \mathcal{X} \rangle$, $i, j = 1, \dots, p$, such that $\partial(a\eta_i) = a_{(1)}p_{j,i}(a_{(2)})\eta_j$ for $i = 1, \dots, p$. A left-invariant differential operator ∂ on \mathcal{C} is called an *m^{th} order differential operator* if $\deg p_{i,j} \leq m$ for all $i, j = 1, \dots, p$ with respect to the given filtration of $\langle \mathcal{X} \rangle$.

Let (Γ, d) be a FODC over \mathcal{A} , $\text{Cl}(\Gamma, \sigma, g)$ a corresponding quantum Clifford algebra and \mathcal{S} a spinor module. Let ∇ and ∇' be linear left connections on $\text{Cl}(\Gamma, \sigma, g)$ and \mathcal{S} , respectively, which are compatible with the multiplication of the quantum Clifford algebra. Then we define the Dirac operator D on $\text{Cl}(\Gamma, \sigma, g)$ by $D := m_{\text{Cl}}\nabla$ and on \mathcal{S} by $D := m_{\text{Cl}, \mathcal{S}}\nabla'$. Equation (20) gives that the Dirac operators on $\text{Cl}(\Gamma, \sigma, g)$ and on \mathcal{S} are first order left-invariant differential operators. At the end of this section it will be shown that the connection Laplacian on \mathcal{S} is a second order differential operator.

Lemma 5.1. *For all $a, b \in \mathcal{A}$ and $X \in \mathcal{X}$ the equation*

$$h(a_{(1)}X(a_{(2)})b^*) = h(a(b_{(1)}S^2(X)^*(b_{(2)}))^*) \quad (41)$$

holds.

Proof. Since both sides of the equation are linear in X it suffices to prove the lemma for $X = X_i$, $i = 1, \dots, \dim \mathcal{X}$. We obtain

$$\begin{aligned} h(a_{(1)}b^*)X_i(a_{(2)}) &= h(a_{(1)}b_{(1)}^*)X_k(a_{(2)})f_l^k(b_{(2)}^*)S(f_i^l)(b_{(3)}^*) \\ &= h(a_{(1)}b_{(1)}^*)(X_l(a_{(2)}b_{(2)}^*) - \varepsilon(a_{(2)})X_l(b_{(2)}^*))S(f_i^l)(b_{(3)}^*) \\ &= (hX_l)(ab_{(1)}^*)S(f_i^l)(b_{(2)}^*) + h(ab_{(1)}^*)S(X_i)(b_{(2)}^*). \end{aligned}$$

Since the Haar functional is right-invariant, we have $hX_l(c) = h(c)X_l(1) = 0$ for $c \in \mathcal{A}$ and therefore the first summand of the last expression vanishes. On the other hand, formula (5) gives

$$S(X_i)(b^*) = \overline{S^2(X_i)^*(b)}$$

from which the assertion follows. ■

Suppose that $\langle \cdot, \cdot \rangle : \mathcal{S} \otimes \mathcal{S}^c \rightarrow \mathbb{C}$ is a hermitean metric on \mathcal{S} . Recall that β is an involution of the algebra $\text{Cl}(\Gamma, \sigma, g)$.

Proposition 5.2. *The Dirac operator D on \mathcal{S} is symmetric with respect to the metric $\langle \cdot, \cdot \rangle$ if and only if $\langle D\varphi, \psi \rangle = \langle \varphi, D\psi \rangle$ for any $\varphi, \psi \in \mathcal{S}_L$ and equation (19) is fulfilled for $u \in \text{Cl}(\Gamma, \sigma, g)$ and $\varphi, \psi \in \mathcal{S}$.*

Proof. Let $a, b \in \mathcal{A}$ and $\varphi, \psi \in \mathcal{S}_L$. Then

$$\begin{aligned} \langle D(a\varphi), b\psi \rangle - \langle a\varphi, D(b\psi) \rangle &= \\ &= \langle a_{(1)}X_i(a_{(2)})\theta_i\varphi + aD\varphi, b\psi \rangle - \langle a\varphi, b_{(1)}X_j(b_{(2)})\theta_j\psi + bD\psi \rangle \\ &= h(a_{(1)}b^*)X_i(a_{(2)})\langle \theta_i\varphi, \psi \rangle - h(ab_{(1)}^*)\overline{X_j(b_{(2)})}\langle \varphi, \theta_j\psi \rangle \\ &\quad + h(ab^*)(\langle D\varphi, \psi \rangle - \langle \varphi, D\psi \rangle) \end{aligned} \quad (*)$$

by Definition 3.2 and formulas (20) and (4). Applying Lemma 5.1 and using formula (5) it follows that (*) is equal to

$$\begin{aligned} &h(ab_{(1)}^*)\overline{B_j^i X_j(b_{(2)})}\langle \theta_i\varphi, \psi \rangle - h(ab_{(1)}^*)\overline{X_j(b_{(2)})}\langle \varphi, \theta_j\psi \rangle \\ &\quad + h(ab^*)(\langle D\varphi, \psi \rangle - \langle \varphi, D\psi \rangle) \\ &= h(ab_{(1)}^*)\overline{B_j^i X_j(b_{(2)})}(\langle \theta_i\varphi, \psi \rangle - \langle \varphi, \beta(\theta_i)\psi \rangle) + h(ab^*)(\langle D\varphi, \psi \rangle - \langle \varphi, D\psi \rangle). \end{aligned} \quad (**)$$

Suppose that $\langle D\varphi, \psi \rangle = \langle \varphi, D\psi \rangle$ for any $\varphi, \psi \in \mathcal{S}_L$ and (19) is fulfilled. Since any element of \mathcal{S} is a linear combination of elements $a\varphi$, $a \in \mathcal{A}$, $\varphi \in \mathcal{S}_L$, we obtain $\langle D\varphi, \psi \rangle = \langle \varphi, D\psi \rangle$ for any $\varphi, \psi \in \mathcal{S}$ by (**).

For the other direction of the assertion of the lemma we suppose that $\langle D(a\varphi), b\psi \rangle - \langle a\varphi, D(b\psi) \rangle = 0$ for $a, b \in \mathcal{A}$ and $\varphi, \psi \in \mathcal{S}_L$. Then trivially $\langle D\varphi, \psi \rangle = \langle \varphi, D\psi \rangle$ for any $\varphi, \psi \in \mathcal{S}_L$. Moreover, from $(**) = 0$ we obtain

$$h(ab_{(1)}^*)\overline{B_j^i X_j(b_{(2)})}(\langle \theta_i\varphi, \psi \rangle - \langle \varphi, \beta(\theta_i)\psi \rangle) = 0$$

for $a, b \in \mathcal{A}$ and $\varphi, \psi \in \mathcal{S}_L$. Since the Haar functional is left-regular, we conclude that $(b_{(1)}B_j^i X_j(b_{(2)}))^*(\langle \theta_i\varphi, \psi \rangle - \langle \varphi, \beta(\theta_i)\psi \rangle) = 0$ for any $b \in \mathcal{A}$ and $\varphi, \psi \in \mathcal{S}_L$. Evaluating ε on this expression and setting $b := b_k$, where $X_i(b_k) = \delta_k^i$, we obtain $\langle \theta_k\varphi, \psi \rangle - \langle \varphi, \beta(\theta_k)\psi \rangle = 0$ for any $\varphi, \psi \in \mathcal{S}_L$. Since the elements of \mathcal{A} and the set $\{\theta_k\}$ generate $\text{Cl}(\Gamma, \sigma, g)$, (19) holds for any $u \in \text{Cl}(\Gamma, \sigma, g)$ and $\varphi, \psi \in \mathcal{S}_L$. Finally, by Remark 4 after Definition 3.2 we have

$$\langle u(a\varphi), b\psi \rangle = \langle (\beta(b)ua)\varphi, \psi \rangle = \langle \varphi, \beta(a)\beta(u)b\psi \rangle = \langle a\varphi, \beta(u)b\psi \rangle$$

for any $a, b \in \mathcal{A}$, $\varphi, \psi \in \mathcal{S}_L$ and $u \in \text{Cl}(\Gamma, \sigma, g)$. ■

Proposition 5.3. *The connection Laplacian $\nabla^*\nabla_{\mathcal{S}}$ associated to the connection $\nabla_{\mathcal{S}}$ on \mathcal{S} is a second order left-invariant differential operator on \mathcal{S} .*

Proof. By the defining equation (40),

$$\langle \nabla^*\nabla_{\mathcal{S}}(a\psi), b\psi' \rangle = -hg(\langle \nabla_{\mathcal{S}}(a\psi), b_{(1)}X_k(b_{(2)})\theta_k \otimes_{\mathcal{A}} \psi' + b\nabla_{\mathcal{S}}(\psi') \rangle_0) \quad (42)$$

for $\psi, \psi' \in \mathcal{S}_L$, $a, b \in \mathcal{A}$. Let us reformulate both summands of the expression on the right hand side. Using (12) and Lemma 5.1 we obtain

$$\begin{aligned} hg(\langle a\theta_i \otimes_{\mathcal{A}} \psi_j, b_{(1)}X_k(b_{(2)})\theta_k \otimes_{\mathcal{A}} \psi' \rangle_0) &= h(a(b_{(1)}X_k(b_{(2)}))^*)g(\theta_i, \theta_k^*)\langle \psi_j, \psi' \rangle \\ &= h(a(b_{(1)}X_k(b_{(2)}))^*)g(-\overline{B_k^l}\theta_l, \theta_i)\langle \psi_j, \psi' \rangle \\ &= -h(a(b_{(1)}B_k^lX_k(b_{(2)}))^*)g(\theta_l, \theta_i)\langle \psi_j, \psi' \rangle \\ &= -h(a_{(1)}X_l(a_{(2)})b^*)g(\theta_l, \theta_i)\langle \psi_j, \psi' \rangle = -\langle a_{(1)}X_l(a_{(2)})g(\theta_l, \theta_i)\psi_j, b\psi' \rangle. \end{aligned}$$

Further, applying (38) we get

$$\begin{aligned} hg(\langle a\theta_i \otimes_{\mathcal{A}} \psi_j, b\nabla_{\mathcal{S}}(\psi') \rangle_0) &= hg(a\theta_i \otimes_{\mathcal{A}} \langle \psi_j, \nabla_{\mathcal{S}}(\psi') \rangle_0 b^*) \\ &= -hg(a\theta_i \otimes_{\mathcal{A}} \langle \nabla_{\mathcal{S}}^* \psi_j, \psi' \rangle_0 b^*) = -hg(\langle a\theta_i \otimes_{\mathcal{A}} \nabla_{\mathcal{S}}^* \psi_j, b\psi' \rangle_0). \end{aligned}$$

From this, equation (42) and since $\langle \cdot, \cdot \rangle$ is non-degenerate we conclude that

$$\begin{aligned} \nabla^*\nabla_{\mathcal{S}}(a\psi) &= a_{(1)}X_kX_l(a_{(2)})g(\theta_k, \theta_l)\psi + a(g \otimes_{\mathcal{A}} \text{id})(\text{id} \otimes_{\mathcal{A}} \nabla_{\mathcal{S}}^*)\nabla_{\mathcal{S}}(\psi) \\ &\quad + a_{(1)}X_k(a_{(2)})(g \otimes_{\mathcal{A}} \text{id})(\theta_k \otimes_{\mathcal{A}} (\nabla_{\mathcal{S}} + \nabla_{\mathcal{S}}^*)(\psi)) \end{aligned} \quad (43)$$

for all $a \in \mathcal{A}$ and $\psi \in \mathcal{S}_L$. ■

In classical differential geometry the connection Laplacian and the Dirac operator (corresponding to the torsion-free euclidean connection) are related by the famous theorem of Bochner [17]. In the present setting, without any requirement on the torsion, only a weak form of this assertion holds. For a stronger result for the quantum group $\text{SL}_q(2)$ see also Theorem 7.5.

Proposition 5.4. *Let $\nabla_{\mathcal{S}}$ be a linear left connection on \mathcal{S} and let D and $\nabla^*\nabla_{\mathcal{S}}$ be the corresponding Dirac operator and connection Laplacian, respectively. Assume that $\ker(\text{id} - \sigma) = \ker(\text{id} - \sigma)^2 (\subset \Gamma \otimes_{\mathcal{A}} \Gamma)$. Then the operator $\nabla^*\nabla_{\mathcal{S}} - D^2$ is a first order left-invariant differential operator on \mathcal{S} .*

Proof. Since $D = m_{\text{cl}, \mathcal{S}} \nabla_{\mathcal{S}}$, from (20) it follows that the operator D^2 acts on \mathcal{S} by the formula

$$\begin{aligned} D^2(a\psi) &= D(a_{(1)}X_j(a_{(2)})\theta_j\psi + aD\psi) \\ &= a_{(1)}X_iX_j(a_{(2)})\theta_i\theta_j\psi + a_{(1)}X_i(a_{(2)})(D(\theta_i\psi) + \theta_iD\psi) + aD^2\psi \end{aligned}$$

for $a \in \mathcal{A}, \psi \in \mathcal{S}$. On the other hand, $\nabla^* \nabla_{\mathcal{S}}$ satisfies the equation (43). Hence by Definition 5.1 we have to show that the mapping $\partial : \mathcal{S} \rightarrow \mathcal{S}$, defined by

$$\partial(a\psi) = a_{(1)}X_iX_j(a_{(2)})(g(\theta_i, \theta_j)\psi - \theta_i\theta_j\psi)$$

is a first order left-invariant differential operator on \mathcal{S} .

Since $\Gamma_L \otimes \Gamma_L$ is a finite dimensional vector space, there exists a polynomial $T \in \mathbb{C}[s]$ in one variable s such that $(\text{id} - \sigma)T(\sigma) = 0$. From the assumption $\ker(\text{id} - \sigma) = \ker(\text{id} - \sigma)^2$ we conclude that $T(1) \neq 0$. Hence there exists a polynomial $T' \in \mathbb{C}[s]$ such that $1 = T(s)/T(1) + (1 - s)T'(s)$. Now we have

$$X_iX_j(a)\theta_i\theta_j\psi = X_iX_j(a)(T(\sigma)/T(1) + (\text{id} - \sigma)T'(\sigma))_{ij}^{kl}\theta_k\theta_l\psi. \quad (*)$$

Definition 3.1 gives that $T(\sigma)_{ij}^{kl}\theta_k\theta_l = T(\sigma)_{ij}^{kl}g(\theta_k, \theta_l) = T(1)g(\theta_i, \theta_j)$ because of $g\sigma = g$. On the other hand, the second requirement on the braiding on page 7 ensures that $X_iX_j(\text{id} - \sigma)_{ij}^{kl} = [X_k, X_l]$ are elements of \mathcal{X} for all k, l . Therefore, equation (*) gives that

$$X_iX_j(a)\theta_i\theta_j\psi = X_iX_j(a)g(\theta_i, \theta_j)\psi + [X_m, X_n]T'(\sigma)_{mn}^{kl}\theta_k\theta_l\psi.$$

This means that ∂ is a first order left-invariant differential operator on \mathcal{S} . ■

6 The quantum group $\text{SL}_q(2)$. An example.

Let \mathcal{A} denote the Hopf algebra $\mathcal{O}(\text{SL}_q(2))$ with generators u_j^i . We use the symbols $\hat{R} = (\hat{R}_{kl}^{ij})$, $\check{C} = (\check{C}^{ij})$ and $\hat{C} = (\hat{C}_{ij})$ for the matrices with entries

$$\begin{aligned} \hat{R}_{kl}^{ij} &= \delta_l^i \delta_k^j q^{\delta_j^i - 1/2} + q^{-1/2} \hat{q} \delta_1^i \delta_1^k \delta_2^j \delta_2^l, \\ \check{C}^{ij} &= q^{-1/2} \delta_1^i \delta_2^j - q^{1/2} \delta_2^i \delta_1^j, \quad \hat{C}_{ij} = -\check{C}^{ij}, \end{aligned} \quad (44)$$

$i, j, k, l = 1, 2$. Further notations are collected in Appendix A.

Let (Γ, d) be one of the $(S^2\text{-invariant bicovariant})$ $4D_{\pm}$ -calculi over \mathcal{A} . There is a basis $\{\theta_{ij} \mid i, j = 1, 2\}$ of Γ_L such that $\Delta_R(\theta_{ij}) = \theta_{kl} \otimes u_i^k u_j^l$ and $\theta_{ij} \triangleleft a := S(a_{(1)})\theta_{ij}a_{(2)} = f_{kl}^{ij}(a)\theta_{kl}$, where

$$f_{kl}^{ij} = \varepsilon_{\pm} \begin{pmatrix} 1 & 0 & q^{-1/2} \hat{q} F K & 0 \\ -q^{1/2} \hat{q} K^{-1} E & K^{-2} & -\hat{q}^2 F E & q^{-1/2} \hat{q} F K^{-1} \\ 0 & 0 & K^2 & 0 \\ 0 & 0 & -q^{1/2} \hat{q} K E & 1 \end{pmatrix}. \quad (45)$$

Equivalently, the right multiplication of the differential 1-forms θ_{ij} by the generators u_l^k of \mathcal{A} are given by

$$\theta_{ij} u_l^k = \nu_0 \hat{R}_{it}^{mr} \hat{R}_{jl}^{-1ts} u_m^k \theta_{rs}, \quad (46)$$

where $\nu_0 = \pm$, depending on the sign of the differential calculus. The element $\theta := q^{1/2} \check{C}^{ij} / \hat{q} \theta_{ij}$ of Γ is biinvariant and the differential $d : \mathcal{A} \rightarrow \Gamma$ can be defined by $da := \theta a - a\theta$ for any $a \in \mathcal{A}$. The dual basis to $\{\theta_{ij}\}$ of the quantum tangent space \mathcal{X} is

$$\begin{aligned} \{X_{11} = -q^{1/2} \varepsilon_{\pm} K^{-1} E, X_{12} = 1/\hat{q} (\varepsilon_{\pm} K^{-2} - \varepsilon), \\ X_{21} = -\hat{q} \varepsilon_{\pm} F E - q/\hat{q} (\varepsilon_{\pm} K^2 - \varepsilon), X_{22} = q^{-1/2} \varepsilon_{\pm} F K^{-1}\} \end{aligned} \quad (47)$$

For these functionals we have

$$S^2(X_{11}) = q^2 X_{11}, \quad S^2(X_{12}) = X_{12}, \quad S^2(X_{21}) = X_{21}, \quad S^2(X_{22}) = q^{-2} X_{22}. \quad (48)$$

We consider three involutions $*$ of \mathcal{A} . Let \dagger denote the involution of $\mathcal{O}(\text{SU}_q(2))$, \star the one of $\mathcal{O}(\text{SU}_q(1,1))$ and \sharp the one of $\mathcal{O}(\text{SL}_q(2, \mathbb{R}))$. In all three cases the involution of \mathcal{A} can be uniquely extended to an involution $*$ of Γ such that $d(a^*) = (da)^*$ for any $a \in \mathcal{A}$. Then the involution of \mathcal{X} is given by

$$\begin{aligned} X_{11}^{\dagger} &= -q X_{22}, \quad X_{12}^{\dagger} = X_{12}, \quad X_{21}^{\dagger} = X_{21}, \quad X_{22}^{\dagger} = -q^{-1} X_{11}, \\ X_{11}^{\star} &= q X_{22}, \quad X_{12}^{\star} = X_{12}, \quad X_{21}^{\star} = X_{21}, \quad X_{22}^{\star} = q^{-1} X_{11}, \\ X_{11}^{\sharp} &= X_{11}, \quad X_{12}^{\sharp} = -X_{12}, \quad X_{21}^{\sharp} = -X_{21} - \hat{q} X_{12}, \quad X_{22}^{\sharp} = q^2 X_{22}. \end{aligned} \quad (49)$$

Hence the matrices $B = (B_{kl}^{ij})$, defined by $S^2(X_{ij})^* = B_{kl}^{ij} X_{kl}$, take for the involutions \dagger, \star and \sharp the form

$$\begin{pmatrix} 0 & 0 & 0 & -q^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -q^{-3} & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & q^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q^{-3} & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} q^{-2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -\hat{q} & -1 & 0 \\ 0 & 0 & 0 & q^4 \end{pmatrix}, \quad (50)$$

respectively. For the involution of Γ we obtain the formulas

$$\begin{aligned} \theta_{11}^{\dagger} &= q^{-1} \theta_{22}, \quad \theta_{12}^{\dagger} = -\theta_{12}, \quad \theta_{21}^{\dagger} = -\theta_{21}, \quad \theta_{22}^{\dagger} = q \theta_{11}, \\ \theta_{11}^{\star} &= -q^{-1} \theta_{22}, \quad \theta_{12}^{\star} = -\theta_{12}, \quad \theta_{21}^{\star} = -\theta_{21}, \quad \theta_{22}^{\star} = -q \theta_{11}, \\ \theta_{11}^{\sharp} &= -\theta_{11}, \quad \theta_{12}^{\sharp} = \theta_{12} - \hat{q} \theta_{21}, \quad \theta_{21}^{\sharp} = \theta_{21}, \quad \theta_{22}^{\sharp} = -q^{-2} \theta_{22}. \end{aligned} \quad (51)$$

Recall that the braiding σ of Γ (defined by (10)) is given by the matrix

$$\sigma_{mn,rs}^{ij,kl} = f_{kl}^{mn}(u_r^i u_s^j) = (\hat{R}_{23} \hat{R}_{12} \hat{R}_{34}^{-1} \hat{R}_{23}^{-1})_{mnrs}^{ijkl} \quad (\text{leg numbering}) \quad (52)$$

and the setting $\sigma(\theta_{mn} \otimes_{\mathcal{A}} \theta_{rs}) = \sigma_{mn,rs}^{ij,kl} \theta_{ij} \otimes_{\mathcal{A}} \theta_{kl}$.

Lemma 6.1. *A linear mapping $g : \Gamma \otimes_{\mathcal{A}} \Gamma \rightarrow \mathcal{A}$ is a bicovariant σ -metric of Γ if and only if it is of the form*

$$g(\theta_{ij} \otimes_{\mathcal{A}} \theta_{kl}) = -q^{1/2} c_1 \hat{C}_{im} \hat{C}_{nl} \hat{R}_{jk}^{-1mn}, \quad c_1 \in \mathbb{R}^\times. \quad (53)$$

Proof. Usual methods (see e.g. [13]) give that g is a bicovariant mapping if and only if $g(\theta_{ij} \otimes_{\mathcal{A}} \theta_{kl}) = \lambda \hat{C}_{ij} \hat{C}_{kl} + \mu \hat{C}_{jk} \hat{C}_{il}$, $\lambda, \mu \in \mathbb{C}$. Equation (46) implies that g is of the form (53) with $c_1 \in \mathbb{C}$. Using (51), from the compatibility with the involution we conclude in all three cases that $c_1 \in \mathbb{R}$. Finally, non-degeneracy of g gives that $c_1 \in \mathbb{R}^\times$. The proof of the property $g\sigma = g$ is an easy computation. The other requirements are fulfilled for each homomorphism of the the bicovariant \mathcal{A} -bimodules $\Gamma \otimes_{\mathcal{A}} \Gamma$ and \mathcal{A} (see [15]). \blacksquare

Let us now fix such a σ -metric g . The nonzero entries of the matrix $(g_{ij,kl})$, $g_{ij,kl} := g(\theta_{ij} \otimes_{\mathcal{A}} \theta_{kl})$ are

$$g_{11,22} = -c_1, \quad g_{12,21} = c_1, \quad g_{21,12} = c_1, \quad g_{22,11} = -q^2 c_1, \quad g_{12,12} = \hat{q} c_1. \quad (54)$$

By Definition 3.1, the quantum Clifford algebra $\text{Cl}(\Gamma, \sigma, g)$ is generated by the following set of relations:

$$\begin{aligned} & \{\theta_{11}^2, \theta_{12}\theta_{11} + q^2\theta_{11}\theta_{12} + \hat{q}\theta_{11}\theta_{21}, \theta_{21}\theta_{11} + \theta_{11}\theta_{21}, \theta_{22}\theta_{11} + \theta_{11}\theta_{22} + (q^2 + 1)c_1, \\ & \theta_{12}^2 + \hat{q}\theta_{11}\theta_{22}, \theta_{21}\theta_{12} + \theta_{12}\theta_{21} + q^{-1}\hat{q}\theta_{11}\theta_{22} - (1 + q^{-2})c_1, \\ & \theta_{22}\theta_{12} + q^2\theta_{12}\theta_{22} + \hat{q}\theta_{21}\theta_{22}, \theta_{21}^2, \theta_{22}\theta_{21} + \theta_{21}\theta_{22}, \theta_{22}^2\}. \end{aligned} \quad (55)$$

Setting $|\theta_{ij}| = 3 - i - j$ and $|u_l^k| = k - l$, from equations (46) and (55) we directly obtain that the algebra $\text{Cl}(\Gamma, \sigma, g)$ becomes a graded algebra with grading $|\cdot|$.

Proposition 6.2. *The quantum Clifford algebra $\text{Cl}(\Gamma, \sigma, g)$ is a simple left-covariant algebra. The elements*

$$\begin{aligned} \psi_1^+ &:= \frac{q}{[2]_{c_1}} \theta_{11} \theta_{12} \theta_{21} \theta_{22}, & \psi_2^+ &:= \theta_{21} \theta_{22}, \\ \psi_1^- &:= \theta_{11} \theta_{21} \theta_{22}, & \psi_2^- &:= q \theta_{12} \theta_{21} \theta_{22} \end{aligned} \quad (56)$$

generate a minimal left-covariant left ideal \mathcal{S}_0 of $\text{Cl}(\Gamma, \sigma, g)$.

The left-covariant left ideals $\mathcal{A}\psi_1^\eta \oplus \mathcal{A}\psi_2^\eta$, $\eta \in \{+, -\}$, are denoted by \mathcal{S}_0^η .

Proof of the Proposition. For the first assertion it is enough to show that there are no non-trivial left-covariant ideals \mathcal{J} of $\text{Cl}(\Gamma, \sigma, g)$.

Suppose that \mathcal{J} is a nonzero left-covariant ideal of $\text{Cl}(\Gamma, \sigma, g)$. Then \mathcal{J} is a free left \mathcal{A} -module with a left-invariant basis. Since $\text{Cl}(\Gamma, \sigma, g)$ is finite dimensional, \mathcal{J} is so, too. Let ρ be a nonzero left-invariant element of \mathcal{J} . We show that then $1 \in \mathcal{J}$, from which $\mathcal{J} = \text{Cl}(\Gamma, \sigma, g)$ follows.

1. If $\theta_{11}\rho = 0$ then $\rho = \theta_{11}\rho'$ for a $\rho' \in \text{Cl}(\Gamma, \sigma, g)$. Otherwise $\theta_{11}\rho \neq 0$. Hence there exists $\rho_1 \in \mathcal{J} \setminus \{0\}$ such that $\rho_1 = \theta_{11}\rho'$.
2. If $\rho_1\theta_{11} = 0$ then $|\rho_1| = 1$. Otherwise $\rho_1\theta_{11} \neq 0$ and we get $|\rho_1\theta_{11}| = 1$. Hence there exists $\rho_2 \in \mathcal{J} \setminus \{0\}$ such that $|\rho_2| = 1$.
3. Let $\rho_2 \in \mathcal{J} \setminus \{0\}$, $|\rho_2| = 1$. Then ρ_2 is a linear combination of the elements θ_{11} , $\theta_{11}\theta_{12}$, $\theta_{11}\theta_{21}$ and $\theta_{11}\theta_{12}\theta_{21}$. If $\rho_2\theta_{21} = 0$ then there are $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\rho_3 := \rho_2 = \theta_{11}(\lambda_1 + \lambda_2\theta_{12})\theta_{21}$. Otherwise $\rho_2\theta_{21} \neq 0$ and there are $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\rho_3 := \rho_2\theta_{21} = \theta_{11}(\lambda_1 + \lambda_2\theta_{12})\theta_{21}$. Again $\rho_3 \in \mathcal{J} \setminus \{0\}$.
4. We have $\theta_{21}\rho_3 = -\lambda_2(1 + q^{-2})c_1\theta_{11}\theta_{21}$. If $\theta_{21}\rho_3 = 0$ then $\lambda_2 = 0$ and hence ρ_3 is a nonzero multiple of $\theta_{11}\theta_{21}$. Otherwise $\theta_{21}\rho_3 \neq 0$ and therefore $\theta_{21}\rho_3$ is a nonzero multiple of $\theta_{11}\theta_{21}$.
5. We obtained that $\rho_4 := \theta_{11}\theta_{21} \in \mathcal{J}$. Then $\mathcal{J} \ni \theta_{12}\rho_4 = -q^2\theta_{11}\theta_{12}\theta_{21}$. We also have $\mathcal{J} \ni \rho_4\theta_{12} = -\theta_{11}\theta_{12}\theta_{21} + (1 + q^{-2})c_1\theta_{11}$, hence $\theta_{11} \in \mathcal{J}$. Therefore $\theta_{11}\theta_{22} \in \mathcal{J}$ and $\mathcal{J} \ni \theta_{22}\theta_{11} = -\theta_{11}\theta_{22} - (1 + q^2)c_1$. Hence $1 \in \mathcal{J}$.

Now we turn to the proof of the second assertion. Using (55) one can easily see that $\text{Cl}(\Gamma, \sigma, g)\mathcal{S}_0 \subset \mathcal{S}_0$. Moreover, since the dimension of the algebra of left-invariant elements of $\text{Cl}(\Gamma, \sigma, g)$ is 16, each minimal left-covariant left ideal of $\text{Cl}(\Gamma, \sigma, g)$ is 4-dimensional. ■

With help of the relations (55) of $\text{Cl}(\Gamma, \sigma, g)$ one can easily determine the left

action of the generators θ_{ij} on \mathcal{S}_0 . We obtain the following table.

	ψ_1^+	ψ_2^+	ψ_1^-	ψ_2^-
θ_{11}	0	ψ_1^-	0	$[2]c_1\psi_1^+$
θ_{12}	$-\hat{q}\psi_1^-$	$q^{-1}\psi_2^-$	$-q[2]c_1\psi_1^+$	0
θ_{21}	$-\psi_1^-$	0	0	$[2]c_1\psi_2^+$
θ_{22}	$-q\psi_2^-$	0	$-q[2]c_1\psi_2^+$	0

Equivalently we can write

$$\theta_{ij}\psi_k^+ = -\hat{R}_{ij}^{lm}\hat{C}_{mk}\psi_l^-, \quad \theta_{ij}\psi_k^- = -q^{1/2}[2]c_1\psi_i^+\hat{C}_{jk}. \quad (57)$$

The following lemma proves that \mathcal{S}_0 is a spinor module of $\text{Cl}(\Gamma, \sigma, g)$.

Lemma 6.3. *The left \mathcal{A} -modules $\mathcal{S}_0, \mathcal{S}_0^+$ and \mathcal{S}_0^- are invariant under right multiplication by \mathcal{A} .*

Proof. By Proposition 6.2, $\mathcal{S}_0 = \text{Cl}(\Gamma, \sigma, g)\psi_2^+$. Since

$$\begin{aligned} \psi_2^+ a &= \theta_{21}\theta_{22}a = a_{(1)}f_{ij}^{21}f_{kl}^{22}(a_{(2)})\theta_{ij}\theta_{kl} \\ &= -q^{1/2}\hat{q}a_{(1)}K^2KE(a_{(2)})\theta_{21}\theta_{21} + a_{(1)}K^2(a_{(2)})\theta_{21}\theta_{22} = a_{(1)}K^2(a_{(2)})\psi_2^+ \end{aligned}$$

for all $a \in \mathcal{A}$, we have $(u\psi_2^+)a = ua_{(1)}K^2(a_{(2)})\psi_2^+ \in \text{Cl}(\Gamma, \sigma, g)\psi_2^+ = \mathcal{S}_0$ for all $u \in \text{Cl}(\Gamma, \sigma, g)$ and $a \in \mathcal{A}$. Further, $\mathcal{S}_0^\eta = \text{Cl}^\eta(\Gamma, \sigma, g)\psi_2^+$ for $\eta \in \{+, -\}$. This implies that $(u\psi_2^\pm)a = ua_{(1)}K^2(a_{(2)})\psi_2^\pm \in \text{Cl}^\eta(\Gamma, \sigma, g)\psi_2^\pm = \mathcal{S}_0^\eta$ for all $u \in \text{Cl}^\eta(\Gamma, \sigma, g)$ and $a \in \mathcal{A}$. \blacksquare

By Lemma 6.3 there exists a matrix $\tilde{f} = (\tilde{f}_j^i)_{i,j=1,\dots,4}$, $\tilde{f}_j^i \in \mathcal{A}^\circ$, such that $\psi_i^+ a = a_{(1)}(\tilde{f}_j^i(a_{(2)})\psi_j^+ + \tilde{f}_{j+2}^i(a_{(2)})\psi_j^-)$ and $\psi_i^- a = a_{(1)}(\tilde{f}_j^{i+2}(a_{(2)})\psi_j^+ + \tilde{f}_{j+2}^{i+2}(a_{(2)})\psi_j^-)$ for any $a \in \mathcal{A}$ and $i = 1, 2$, where the sum is running over $j = 1, 2$. Similarly to the proof of the lemma one can compute the matrix elements \tilde{f}_j^i . Using the commutation relations (55) of $\text{Cl}(\Gamma, \sigma, g)$ and (45) one gets

$$\tilde{f}_j^i = \begin{pmatrix} 1 & q^{-1/2}\hat{q}FK & 0 & 0 \\ 0 & K^2 & 0 & 0 \\ 0 & 0 & \varepsilon_\pm K^2 & 0 \\ 0 & 0 & -q^{-1/2}\hat{q}\varepsilon_\pm KE & \varepsilon_\pm \end{pmatrix}. \quad (58)$$

For the generators u_j^i of \mathcal{A} this means that

$$\psi_i^+ u_k^j = \nu_+ q^{k-3/2} u_l^j \psi_m^+ \hat{R}_{ik}^{lm}, \quad \psi_i^- u_k^j = \nu_- q^{k-3/2} u_l^j \psi_m^- \hat{R}_{ik}^{-1lm}, \quad (59)$$

where the notation $\nu_+ := 1$, $\nu_- := \nu_0 = \pm 1$ (depending on the sign of the differential calculus) is used.

Let χ denote the algebra automorphism of \mathcal{A} defined by $\chi \triangleright u_j^i := q^{j-i} u_j^i$ ($\chi \triangleright a = q^{-|a|} a$ for homogeneous elements $a \in \mathcal{A}$ with respect to the grading $|\cdot|$) and let $\tilde{\mathcal{A}}$ denote the left crossed product algebra $\mathcal{A} \rtimes \mathbb{C}[\chi, \chi^{-1}]$ with multiplication $(a\chi^k)(b\chi^l) = a(\chi^k \triangleright b)\chi^{k+l}$. Then obviously $\tilde{\mathcal{A}}$ is an \mathcal{A} -bimodule. The settings $\Delta(\chi) := \chi \otimes \chi$ and $S(\chi) = \chi^{-1}$ turn $\tilde{\mathcal{A}}$ into a Hopf algebra. Let us define a right coaction $\Delta'_R : (\mathcal{S}_0)_L \rightarrow (\mathcal{S}_0)_L \otimes \tilde{\mathcal{A}}$ on the vector space $(\mathcal{S}_0)_L$ by

$$\Delta'_R(\psi_i^+) = \psi_j^+ \otimes u_i^j \chi, \quad \Delta'_R(\psi_i^-) = \psi_j^- \otimes u_i^j \chi, \quad i = 1, 2. \quad (60)$$

Recall the definition of a Doi-Hopf module in [8].

Theorem 6.4. (i) *The right coaction Δ'_R on $(\mathcal{S}_0)_L$ can be extended to a right coaction Δ_R on the left-covariant \mathcal{A} -bimodule \mathcal{S}_0 . This means that \mathcal{S}_0 together with the multiplication from the right as a right action of \mathcal{A} and Δ_R as a right coaction of $\tilde{\mathcal{A}}$ becomes also a Doi-Hopf module in the category $\mathbf{M}(\mathcal{A})_{\tilde{\mathcal{A}}}^{\tilde{\mathcal{A}}}$.*
(ii) *The right coaction Δ_R on \mathcal{S}_0 is compatible with the left multiplication of $\text{Cl}(\Gamma, \sigma, g)$ on \mathcal{S}_0 , i. e. $\Delta_R(u\psi) = \Delta_R(u)\Delta_R(\psi)$ for any $u \in \text{Cl}(\Gamma, \sigma, g)$ and $\psi \in \mathcal{S}_0$.*

This theorem gives reason to think about $\tilde{\mathcal{A}}$ as the function algebra of the quantum spin group corresponding to the quantum Clifford algebra $\text{Cl}(\Gamma, \sigma, g)$.

Proof. We set $\Delta_R(a\psi_i^\eta) = a_{(1)}\psi_j^\eta \otimes a_{(2)}u_i^j \chi$ for $a \in \mathcal{A}$, $\eta \in \{+, -\}$ and $i = 1, 2$. Then compatibility of Δ_R with the right action of \mathcal{A} on \mathcal{S}_0 means that $\Delta_R(\psi a) = \Delta_R(\psi)\Delta(a)$. From (59) we conclude that

$$\begin{aligned} \Delta_R(\psi_i^\eta u_k^j) &= \Delta_R(\nu_\eta q^{k-3/2} u_l^j \psi_m^\eta \hat{R}_{ik}^{\eta lm}) = \nu_\eta q^{k-3/2} u_r^j \psi_s^\eta \hat{R}_{ik}^{\eta lm} \otimes u_l^r u_m^s \chi \\ &= \nu_\eta q^{k-3/2} u_r^j \psi_s^\eta \otimes u_i^l u_k^m \hat{R}_{lm}^{\eta rs} \chi = \nu_\eta q^{m-3/2} u_r^j \psi_s^\eta \hat{R}_{lm}^{\eta rs} \otimes q^{k-m} u_i^l u_k^m \chi \\ &= \psi_l^\eta u_m^j \otimes u_i^l \chi u_k^m = \Delta_R(\psi_i^\eta) \Delta(u_k^j). \end{aligned}$$

For the second assertion it suffices to prove the formula $\Delta_R(\theta_{ij}\psi_k^\eta) = \Delta_R(\theta_{ij})\Delta_R(\psi_k^\eta)$ for $\eta \in \{+, -\}$. Using (57) we obtain

$$\begin{aligned} \Delta_R(\theta_{ij})\Delta_R(\psi_k^+) &= \theta_{mn}\psi_l^+ \otimes u_i^m u_j^n u_k^l \chi = -\hat{R}_{mn}^{rs} \hat{C}_{sl} \psi_r^- \otimes u_i^m u_j^n u_k^l \chi \\ &= -\hat{C}_{sl} \psi_r^- \otimes u_m^r u_n^s \hat{R}_{ij}^{mn} u_k^l \chi = -\hat{R}_{ij}^{mn} \psi_r^- \otimes u_m^r \hat{C}_{nk} \chi \\ &= \Delta_R(-\hat{R}_{ij}^{mn} \hat{C}_{nk} \psi_m^-) = \Delta_R(\theta_{ij}\psi_k^+). \end{aligned}$$

The statement for $\eta = -$ can be shown similarly. ■

One can ask whether the introduced $*$ -structures on \mathcal{A} can be extended to the left-covariant \mathcal{A} -bimodule \mathcal{S}_0 . Such an extension is not unique in general. One calls the involutions $*$ and $*'$ of \mathcal{S}_0 *equivalent* if there exists an automorphism φ of the left-covariant \mathcal{A} -bimodule \mathcal{S}_0 , $\varphi(a\psi b) = a\varphi(\psi)b$ for $a, b \in \mathcal{A}, \psi \in \mathcal{S}_0$, such that $\varphi(\psi^*) = \varphi(\psi)^{*'}$ for all $\psi \in \mathcal{S}_0$. To determine the involutions of \mathcal{S}_0 , we use the necessary conditions

$$\psi_i^* = \lambda_j^i \psi_j, \quad \lambda_j^i \in \mathbb{C}, \quad \lambda_j^i \tilde{f}_k^j = (\tilde{f}_l^i)^* \lambda_k^l, \quad i, k = 1, 2, 3, 4, \quad (61)$$

which stem from the compatibility of the involution with the left coaction and the bimodule structure of \mathcal{S}_0 , respectively. Here we used the notation $\psi_i = \psi_i^+$ for $i = 1, 2$ and $\psi_i = \psi_{i-2}^-$ for $i = 3, 4$.

Let us consider the $4D_+$ -calculus. The involutions $(\cdot)^\dagger$ and $(\cdot)^*$ of \mathcal{A} can be extended to involutions of the left-covariant \mathcal{A} -bimodule \mathcal{S}_0 . Moreover, these involutions are unique up to equivalence of left-covariant bimodules. In particular, the formulas

$$\begin{aligned} (\psi_1^+)^\dagger &= -\psi_2^-, (\psi_2^+)^\dagger = \psi_1^-, (\psi_1^-)^\dagger = \psi_2^+, (\psi_2^-)^\dagger = -\psi_1^+, \\ (\psi_1^+)^* &= \psi_2^-, (\psi_2^+)^* = \psi_1^-, (\psi_1^-)^* = \psi_2^+, (\psi_2^-)^* = \psi_1^+ \end{aligned} \quad (62)$$

hold. Observe that the setting $\chi^* := \chi$ together with each of the three introduced involutions of \mathcal{A} turn $\tilde{\mathcal{A}}$ into a Hopf $*$ -algebra. It is easy to see that the involutions \dagger and $*$ of \mathcal{S}_0 are compatible with the right coaction Δ_R of $\tilde{\mathcal{A}}$. On the other hand, if we consider the $4D_-$ -calculus then there exists no involution of the left-covariant \mathcal{A} -bimodule \mathcal{S}_0 extending $(\cdot)^\dagger$ or $(\cdot)^*$ of \mathcal{A} . However, for both calculi the involution $(\cdot)^\sharp$ of \mathcal{A} can be extended to an involution of the left-covariant \mathcal{A} -bimodule \mathcal{S}_0 . This involution is unique up to equivalence of left-covariant \mathcal{A} -bimodules, and also compatible with the right coaction of $\tilde{\mathcal{A}}$. The corresponding formulas read as

$$(\psi_1^+)^\sharp = q^{-1}\psi_1^+, \quad (\psi_2^+)^\sharp = \psi_2^+, \quad (\psi_1^-)^\sharp = \psi_1^-, \quad (\psi_2^-)^\sharp = q\psi_2^-. \quad (63)$$

Theorem 6.5. *For each one of the given involutions $*$ of \mathcal{A} there exists a left-covariant (hermitean non-degenerate) metric $\langle \cdot, \cdot \rangle$ on \mathcal{S}_0 which satisfies (19). Moreover, this metric is unique up to a nonzero real factor c_2 . The explicit formula for the metric is given by*

$$\begin{aligned} \langle \psi_i^\eta, \psi_j^{\eta'} \rangle &= c_2 \delta_{-\eta'}^\eta \delta_{i,j} (\delta_1^i + q^2 \delta_2^i), \\ \langle \psi_i^\eta, \psi_j^{\eta'} \rangle &= c_2 \delta_{-\eta'}^\eta \delta_{i,j} (\delta_1^i - q^2 \delta_2^i), \\ \langle \psi_i^\eta, \psi_j^{\eta'} \rangle &= c_2 \delta_{\eta'}^\eta \delta_{i+j,3} (q^{-1} \delta_{i,1} + q \delta_{i,2}) (\delta_{\eta,+} + [2] c_1 \delta_{\eta,-}) \end{aligned} \quad (64)$$

for the involutions \dagger , \star and \sharp , respectively.

Observe that non of the obtained metrics is positive definite.

Proof. Since $\text{Cl}(\Gamma, \sigma, g)$ is a simple left-covariant algebra by Proposition 6.2, the existence of such a metric follows from Proposition 3.3. To obtain the given formulas for the metric, one should choose the elements $x = -q[2]c_1/c_2\psi_1^-$, $-q[2]c_1/c_2\psi_1^-$ and $-q^{-2}[2]c_1/c_2\psi_2^+$ of $\beta(\mathcal{S})\mathcal{S}$ in case of the involution \dagger , \star and \sharp , respectively.

Now we prove the uniqueness of the metric for the involution \dagger . For the other involutions the proof is analogous. Since \mathcal{S} is a minimal left ideal, each vector $\psi \in \mathcal{S}_L \setminus \{0\}$ is cyclic. Hence because of (19) the numbers $\langle \psi_i^\pm, \psi \rangle$ determine the metric completely. Set $\psi := \psi_2^-$. From the table before equation (57) we obtain that $\theta_{12}\psi_2^- = 0$. Hence $0 = \langle \psi', \theta_{12}\psi_2^- \rangle = \langle \beta(\theta_{12})\psi', \psi_2^- \rangle$. By (50) it follows that $\beta(\theta_{12}) = \theta_{12}$. Setting $\psi' = \psi_1^+$, ψ_2^+ and ψ_1^- , again from the table before (57) we conclude that $\langle \psi_1^-, \psi_2^- \rangle = \langle \psi_2^-, \psi_2^- \rangle = \langle \psi_1^+, \psi_2^- \rangle = 0$. Hermiticity and non-degeneracy of the metric imply that the remaining parameter $\langle \psi_2^+, \psi_2^- \rangle$ is a nonzero real number. \blacksquare

Let us define a right coaction Δ_R of $\mathcal{A} \rtimes \mathbb{C}[\chi, \chi^{-1}]$ on $\mathcal{S}_0 \otimes \mathcal{S}_0^c$ by

$$\Delta_R(\psi \otimes \psi'^c) := (\psi_{(0)} \otimes \psi'_{(0)}{}^c) \otimes \psi_{(1)}\chi^{-2}(\psi'_{(1)})^*, \quad (65)$$

where $*$ is the involution of $\mathcal{A} \rtimes \mathbb{C}[\chi, \chi^{-1}]$. Since $\chi^* = \chi$, the image of Δ_R is a subspace of $\mathcal{S} \otimes \mathcal{S}^c \otimes \mathcal{A}$. It is not difficult to prove that for the involutions \dagger and \star (but not for \sharp) the mappings $\langle \cdot, \cdot \rangle_0 : \mathcal{S}_0 \otimes \mathcal{S}_0^c \rightarrow \mathcal{A}$ and $\langle \cdot, \cdot \rangle : \mathcal{S}_0 \otimes \mathcal{S}_0^c \rightarrow \mathbb{C}$ are right-covariant: $(\langle \cdot, \cdot \rangle_0 \otimes \text{id})\Delta_R = \Delta\langle \cdot, \cdot \rangle_0$ on $\mathcal{S}_0 \otimes \mathcal{S}_0^c$.

Proposition 6.6. *Let g be one of the σ -metrics of Lemma 6.1. Then there exist 8 linear left connections ∇ on Γ compatible with the multiplication of the quantum Clifford algebra $\text{Cl}(\Gamma, \sigma, g)$. They can be given by the formula (29) with one of the mappings $\tilde{\sigma} := \nu\tilde{\sigma}_{(i)}$, $i = 1, 2, 3, 4$, where $\nu \in \{+, -\}$,*

$$\tilde{\sigma}_{(i)}(\theta_{kl} \otimes \theta_{mn}) = (\hat{R}_{23}\hat{R}_{12}^{\eta_i}\hat{R}_{34}^{\eta'_i}\hat{R}_{23}^{-1})_{klmn}^{rstx}\theta_{rs} \otimes \theta_{tx}, \quad (66)$$

and (η_i, η'_i) denotes the pair of signs $(+, +)$, $(+, -)$, $(-, +)$ and $(-, -)$ for $i = 1, 2, 3$ and 4, respectively.

Remark. None of left connections ∇ in Proposition 6.6 satisfies (37). For

example, we have

$$(\text{id} - \sigma)(\text{id} + q\tilde{\sigma}_{(1)}) = 0, \quad (\text{id} - \sigma)(\text{id} + q^{-2}\tilde{\sigma}_{(2)})(\text{id} + q^2\tilde{\sigma}_{(2)}) = 0. \quad (67)$$

■

Proof of the Proposition. We only have to show that the assumptions of Proposition 4.3 are fulfilled. This is very easily done for $\nu\tilde{\sigma}_{(2)} = \nu\sigma$ and $\nu\tilde{\sigma}_{(3)} = \nu\sigma^{-1}$. Let us verify them for $\nu\tilde{\sigma}_{(1)}$. From (46) follows that $\nu\tilde{\sigma}_{(1)}$ defines a homomorphism of the \mathcal{A} -bimodule $\Gamma \otimes_{\mathcal{A}} \Gamma$. Indeed, the coefficient of $u_t^r \theta_{xy} \otimes_{\mathcal{A}} \theta_{zw}$ of the expression $\nu\tilde{\sigma}_{(1)}(\theta_{kl} \otimes_{\mathcal{A}} \theta_{mn})u_s^r - \nu\tilde{\sigma}_{(1)}(\theta_{kl} \otimes_{\mathcal{A}} \theta_{mn})u_s^r$ is

$$(\nu\hat{R}_{12}\hat{R}_{23}^{-1}\hat{R}_{34}\hat{R}_{45}^{-1}\hat{R}_{23}\hat{R}_{12}\hat{R}_{34}\hat{R}_{23}^{-1} - \nu\hat{R}_{34}\hat{R}_{23}\hat{R}_{45}\hat{R}_{34}^{-1}\hat{R}_{12}\hat{R}_{23}^{-1}\hat{R}_{34}\hat{R}_{45}^{-1})_{klmns}^{xyzw},$$

or

in the graphical calculus. This is zero because of the Yang-Baxter equation. Inserting the matrices (52) and (66) into the formulas $\sigma_{23}\tilde{\sigma}_{12}\tilde{\sigma}_{23} = \tilde{\sigma}_{12}\tilde{\sigma}_{23}\sigma_{12}$ and $g_{23}\tilde{\sigma}_{12}\tilde{\sigma}_{23} = g_{12}$, we obtain the requirements

But these formulas are also satisfied. Hence by Proposition 4.3, the linear left connection ∇ on Γ given by $\nu\tilde{\sigma}_{(1)}$, $\nu \in \{1, -1\}$, is compatible with the multiplication of $\text{Cl}(\Gamma, \sigma, g)$. ■

Now let us look for linear left connections $\nabla_{\mathcal{S}}$ on the spinor module \mathcal{S}_0 which are compatible with the right coaction of $\tilde{\mathcal{A}}$ on \mathcal{S}_0 . This means that

∇_S should satisfy the equation $\Delta_R(\nabla_S\psi) = (\nabla_S \otimes \text{id})\Delta_R\psi$ for all $\psi \in \mathcal{S}_0$. Since Γ is an inner FODC, by Lemma 4.2 it suffices to determine all homomorphisms $\tau : \mathcal{S}_0 \otimes_{\mathcal{A}} \Gamma \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{S}_0$ and $V : \mathcal{S}_0 \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{S}_0$ of left-covariant bimodules, which are compatible with Δ_R . Let $V : \mathcal{S}_0^\eta \rightarrow \Gamma \otimes_{\mathcal{A}} \mathcal{S}_0^{\eta'}$, $\eta, \eta' \in \{+, -\}$, be a homomorphism of left-covariant left \mathcal{A} -modules. Hence there exist $V_l^{ijk} \in \mathbb{C}$, $i, j, k, l = 1, 2$, such that $V(\psi_l^\eta) = V_l^{ijk} \theta_{ij} \otimes_{\mathcal{A}} \psi_k^{\eta'}$. Compatibility of V with Δ_R implies that $(V_l^{ijk}) \in \text{Mor}(u, u \otimes u \otimes u)$. Using now (59) and (46), one can determine whether V is a homomorphism of right \mathcal{A} -modules. An equivalent criterion is that the matrix $\bar{V} := (V_l^{ijk})$ satisfies the equation

$$\nu_0 \nu_{\eta'} \hat{R}_{12} \hat{R}_{23}^{-1} \hat{R}_{34}^{\eta'} \bar{V}_1 = \nu_\eta \bar{V}_2 \hat{R}_{12}^{\eta} \quad (\text{leg numbering}). \quad (68)$$

Since $\bar{V} \in \text{Mor}(u, u \otimes u \otimes u)$, this implies that

$$V(\psi_i^+) = \alpha_+ \check{C}^{jk} \theta_{ij} \otimes_{\mathcal{A}} \psi_k^-, \quad V(\psi_i^-) = \alpha_- \hat{R}_{mi}^{kl} \check{C}^{jm} \theta_{jk} \otimes_{\mathcal{A}} \psi_l^+, \quad (69)$$

$\alpha_+, \alpha_- \in \mathbb{C}$. Let \bar{V}_η , $\eta \in \{+, -\}$, denote the complex matrix defined by $V(\psi_l^\eta) = (\bar{V}_\eta)_l^{ijk} \theta_{ij} \otimes_{\mathcal{A}} \psi_k^{-\eta}$.

Similar considerations lead to the determination of all homomorphisms τ of the left-covariant \mathcal{A} -bimodules $\mathcal{S}_0^\eta \otimes_{\mathcal{A}} \Gamma$ and $\Gamma \otimes_{\mathcal{A}} \mathcal{S}_0^{\eta'}$, such that τ is compatible with Δ_R . Since τ is a homomorphism of left-covariant left \mathcal{A} -modules, there exists a complex matrix $\bar{\tau} := (\tau_{ijk}^{rst})$, such that $\tau(\psi_i^\eta \otimes_{\mathcal{A}} \theta_{jk}) = q^{j+k} \tau_{ijk}^{rst} \theta_{rs} \otimes_{\mathcal{A}} \psi_t^{\eta'}$. Compatibility of τ with Δ_R implies that $\bar{\tau} \in \text{Mor}(u \otimes u \otimes u)$. Moreover, τ is a homomorphism of right \mathcal{A} -modules if and only if

$$\nu_\eta \bar{\tau}_{234} \hat{R}_{12}^\eta \hat{R}_{23} \hat{R}_{34}^{-1} = \nu_{\eta'} \hat{R}_{12} \hat{R}_{23}^{-1} \hat{R}_{34}^{\eta'} \bar{\tau}_{123}. \quad (70)$$

The solutions of this equation correspond to the mappings τ given by

$$\begin{aligned} \tau(\psi_i^+ \otimes_{\mathcal{A}} \theta_{jk}) &= q^{j+k} (\gamma_1^+ \delta_i^r \hat{R}_{jk}^{st} + \gamma_2^+ \hat{R}_{lk}^{st} \hat{R}_{ij}^{rl}) \theta_{rs} \otimes_{\mathcal{A}} \psi_t^+, \\ \tau(\psi_i^- \otimes_{\mathcal{A}} \theta_{jk}) &= q^{j+k} (\gamma_1^- \hat{R}_{ij}^{-1rs} \delta_k^t + \gamma_2^- \hat{R}_{lk}^{st} \hat{R}_{ij}^{-1rl}) \theta_{rs} \otimes_{\mathcal{A}} \psi_t^-, \end{aligned} \quad (71)$$

$\gamma_1^\pm, \gamma_2^\pm \in \mathbb{C}$. Let $\bar{\tau}_\eta$, $\eta \in \{+, -\}$, denote the complex matrix defined by $\tau(\psi_i^\eta \otimes_{\mathcal{A}} \theta_{jk}) = q^{j+k} (\bar{\tau}_\eta)_{ijk}^{rst} \theta_{rs} \otimes_{\mathcal{A}} \psi_t^\eta$.

Finally, to obtain a linear left connection on the spinor module \mathcal{S}_0 compatibility with the left action of $\text{Cl}(\Gamma, \sigma, g)$ has to hold. This can be checked by Proposition 4.4. Observe that because of equations (69) and (71), for $\psi \in \mathcal{S}_0^\eta$, $\eta \in \{+, -\}$, the first and second summand of (36) is an element of $\Gamma \otimes_{\mathcal{A}} \mathcal{S}_0^{-\eta}$

and $\Gamma \otimes_{\mathcal{A}} \mathcal{S}_0^\eta$, respectively. Hence both summands have to vanish. Inserting formula (66) for $\tilde{\sigma}$ and equation (57) and using the graphical calculus the following equations have to hold.

$$-q^{1/2}[2]c_1\nu \begin{array}{c} \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} = 0, \quad (72)$$

$$-\nu \begin{array}{c} \text{Diagram 3} \end{array} + q^{1/2}[2]c_1 \begin{array}{c} \text{Diagram 4} \end{array} = 0, \quad (73)$$

$$- \begin{array}{c} \text{Diagram 5} \end{array} + \nu \begin{array}{c} \text{Diagram 6} \end{array} = 0, \quad (74)$$

$$-q^{1/2}[2]c_1 \begin{array}{c} \text{Diagram 7} \end{array} + q^{1/2}[2]c_1\nu \begin{array}{c} \text{Diagram 8} \end{array} = 0, \quad (75)$$

where $i \in \{1, 2, 3, 4\}$ and $\nu \in \{+, -\}$. We obtain the solutions $V = 0$ and

$$\begin{aligned} \tau(\psi_m^+ \otimes_{\mathcal{A}} \theta_{jk}) &= \gamma q^{j+k} \hat{R}_{lk}^{st} \hat{R}_{mj}^{\eta_i rl} \theta_{rs} \otimes_{\mathcal{A}} \psi_t^+, \\ \tau(\psi_m^- \otimes_{\mathcal{A}} \theta_{jk}) &= \nu \gamma q^{j+k} \hat{R}_{lk}^{\eta_i' st} \hat{R}_{mj}^{-1 rl} \theta_{rs} \otimes_{\mathcal{A}} \psi_t^-, \end{aligned} \quad (76)$$

$\gamma \in \mathbb{C}$. These considerations prove the first part of the following theorem.

Theorem 6.7. *Let g be one of the σ -metrics of Lemma 6.1 and let ∇ be one of the left connections in Proposition 6.6. Then there exists a 1-parameter family of linear left connections ∇_S on the spinor module \mathcal{S}_0 . This connections are given by (29) with $V = 0$ and $\tilde{\sigma} := \tau$ as in (76). Moreover,*

- (i) the left-covariant \mathcal{A} -subbimodules \mathcal{S}_0^η of \mathcal{S}_0 are invariant with respect to ∇_S in the sense that $\nabla_S \mathcal{S}_0^\eta \subseteq \Gamma \otimes_{\mathcal{A}} \mathcal{S}_0^\eta$.
- (ii) the Dirac operator D corresponding to ∇_S is symmetric with respect to the metric $\langle \cdot, \cdot \rangle$ given in Theorem 6.5 if and only if $q \in \mathbb{R}$ and $\bar{\gamma} = \gamma$, or $|q| = 1$ and $\bar{\gamma} = q^{6-3/2\eta_i-3/2\eta'_i} \nu \gamma$.

For some computations it is necessary to know ∇_S in the more explicit form

$$\begin{aligned}\nabla_S \psi_m^+ &= \left(\frac{q^{1/2}}{\hat{q}} (1 - q^{7/2-\eta_i/2} \gamma) \check{C}^{jk} \delta_m^l - q^{5/2} \frac{1-\eta_i}{2} \gamma \delta_m^j \check{C}^{kl} \right) \theta_{jk} \otimes_{\mathcal{A}} \psi_l^+, \\ \nabla_S \psi_m^- &= \left(\frac{q^{1/2}}{\hat{q}} (1 - q^{7/2+\eta'_i/2} \nu \gamma) \check{C}^{jk} \delta_m^l - q^{5/2} \frac{1+\eta'_i}{2} \nu \gamma \delta_m^j \check{C}^{kl} \right) \theta_{jk} \otimes_{\mathcal{A}} \psi_l^-. \end{aligned} \quad (77)$$

Proof of (ii). Because of Proposition 5.2 we only have to show that $\langle D\varphi, \psi \rangle = \langle \varphi, D\psi \rangle$ for all $\varphi, \psi \in (\mathcal{S}_0)_L$. From (76) and (57) we easily compute that

$$D\psi_j^+ = \frac{q^{-1}}{\hat{q}} (1 - q^{9/2-3/2\eta_i} \gamma) \psi_j^-, \quad D\psi_j^- = \frac{-q[2]c_1}{\hat{q}} (1 - q^{3/2-3/2\eta'_i} \nu \gamma) \psi_j^+. \quad (78)$$

Inserting this into (64) we obtain that D is symmetric if and only if γ satisfies the given condition. \blacksquare

7 Invariant differential operators on \mathcal{S}_0

In Section 5 it was shown that the Dirac operator and the connection Laplacian on \mathcal{S}_0 are left-invariant differential operators. In this section we will prove that they are compatible with the right coaction Δ_R of $\tilde{\mathcal{A}}$ on \mathcal{S}_0 , i.e. $\Delta_R(D\psi) = (D \otimes \text{id})\Delta_R(\psi)$ and $\Delta_R(\nabla^* \nabla_S \psi) = (\nabla^* \nabla_S \otimes \text{id})\Delta_R(\psi)$ for $\psi \in \mathcal{S}_0$. Such left-invariant differential operators are called *invariant*. On the other hand, sums, complex multiples and products of invariant differential operators ∂_1 and ∂_2 on \mathcal{S}_0 ,

$$(\partial_1 + \partial_2)\psi = \partial_1\psi + \partial_2\psi, \quad (\lambda\partial_1)\psi = \lambda(\partial_1\psi), \quad (\partial_1\partial_2)\psi = \partial_1(\partial_2\psi), \quad (79)$$

where $\lambda \in \mathbb{C}$, are again invariant differential operators on \mathcal{S}_0 . Hence the set of invariant differential operators on \mathcal{S}_0 forms an algebra $\mathcal{D}(\mathcal{S}_0)$. Now we want to find out whether the Dirac operator and the connection Laplacian are generic in the algebra $\mathcal{D}(\mathcal{S}_0)$ in some sense.

Let ∂ be a left-invariant differential operator on \mathcal{S}_0 ,

$$\partial(a\psi_i^\eta) = \sum_{j,k,\eta'} a_{(1)} p_{k,j}^{\eta,\eta'}(a_{(2)}) \hat{C}_{ji} \psi_k^{\eta'}. \quad (80)$$

Then ∂ is invariant if and only if $\Delta_R \partial(a\psi_i^\eta) = (\partial \otimes \text{id}) \Delta_R(a\psi_i^\eta)$ for $\eta \in \{+, -\}$, $i = 1, 2$, $a \in \mathcal{A}$. This is equivalent to the equation

$$a_{(1)} \otimes a_{(2)} p_{k,j}^{\eta,\eta'}(a_{(3)}) \hat{C}_{ji} u_k^l = a_{(1)} \otimes p_{l,k}^{\eta,\eta'}(a_{(2)}) \hat{C}_{kj} a_{(3)} u_i^j, \quad (81)$$

$a \in \mathcal{A}$, $i, l = 1, 2$, $\eta, \eta' \in \{+, -\}$. Multiplying by $S(u_m^i)$ from the right, applying $S \otimes \text{id}$ and then multiplying both factors of the tensor product we obtain

$$p_{k,j}^{\eta,\eta'}(a) \hat{C}_{ji} u_k^l S(u_m^i) = S(a_{(1)}) p_{l,j}^{\eta,\eta'}(a_{(2)}) \hat{C}_{jm} a_{(3)}, \quad (82)$$

$a \in \mathcal{A}$, $l, m = 1, 2$, $\eta, \eta' \in \{+, -\}$. In particular, ∂ is the sum of four invariant differential operators $\partial_{\eta,\eta'} : \mathcal{S}_0 \rightarrow \mathcal{S}_0^{\eta'}$, $\eta, \eta' \in \{+, -\}$, such that $\partial_{\eta,\eta'}(\psi) = P_{\eta'}(\partial(\psi))$ (projection to the component in $\mathcal{S}_0^{\eta'}$ with respect to the decomposition $\mathcal{S}_0 = \mathcal{S}_0^+ \oplus \mathcal{S}_0^-$) for $\psi \in \mathcal{S}_0^\eta$ and $\partial_{\eta,\eta'}(\psi) = 0$ for $\psi \in \mathcal{S}_0^{-\eta}$. From now on we suppose that $\partial = \partial_{+,+}$. The results carry over to the other cases as well.

By Definition 5.1 the functionals $p_{i,j} = p_{i,j}^{+,+}$ are elements of the unital subalgebra $\langle \mathcal{X}_\pm \rangle$ of $U_q(\mathfrak{sl}_2)$ generated by X_{kl} , $k, l = 1, 2$. Evaluating a functional $f \in \mathcal{A}^\circ$ on both sides of equation (82) yields that

$$\begin{aligned} (\text{ad}_R(f) p_{l,n})(a) &:= (S(f_{(1)}) p_{l,j} f_{(2)})(a) \hat{C}_{jm} \check{C}^{mn} \\ &= p_{k,j}(a) \hat{C}_{ji} f(u_k^l \check{C}^{ir} \hat{C}_{sm} u_r^s) \check{C}^{mn} = f(u_k^l u_r^n) p_{k,r}(a) \end{aligned} \quad (83)$$

for all $l, n = 1, 2$, $f \in \mathcal{A}^\circ$, $a \in \mathcal{A}$. Moreover, since \mathcal{A}° separates the elements of \mathcal{A} (see [16, Theorem 11.22]), equation (83) is equivalent to (82).

Let $I(\partial)$ denote the pair $(v_0^0, (v_{-1}^1, v_0^1, v_1^1))$, where

$$v_k^m := \frac{-1}{[2]^{1/2}} \sum_{i,j} C_q^{-1}(1/2, 1/2, m; i - 3/2, j - 3/2, k) p_{i,j}, \quad (84)$$

$m = 0, 1$, $k = -m, \dots, m$, and $C_q^{-1}(l_1, l_2, l; i, j, k)$ are the inverse Clebsch-Gordan coefficients of the tensor product of the corepresentations $\mathbf{u}^{(l_1)}$ and $\mathbf{u}^{(l_2)}$ (see Appendix A). Then

$$\mathbf{u}_{mn}^{(l)} C_q^{-1}(l_1, l_2, l; r, s, n) = \sum_{i,j} C_q^{-1}(l_1, l_2, l; i, j, m) \mathbf{u}_{ir}^{(l_1)} \mathbf{u}_{js}^{(l_2)} \quad (85)$$

and (83) implies that

$$\mathrm{ad}_R(f)v_0^0 = f(1)v_0^0, \quad \mathrm{ad}_R(f)v_i^1 = f(\mathbf{u}_{ij}^{(1)})v_j^1 \quad (86)$$

for all $f \in \mathcal{A}^\circ$, $i = 1, 2, 3$.

Lemma 7.1. *The mapping I gives a one-to-one correspondence between invariant differential operators ∂ on \mathcal{S}_0^+ and pairs $v = (v_0^0, (v_{-1}^1, v_0^1, v_1^1))$, where $v_0^0, v_i^1 \in \langle \mathcal{X}_\pm \rangle$ satisfy (86). Under this correspondence we have*

$$I(\mathrm{id}) = (\varepsilon, (0, 0, 0)), \quad I(\lambda\partial + \lambda'\partial') = \lambda I(\partial) + \lambda' I(\partial') \quad (87)$$

for $\lambda, \lambda' \in \mathbb{C}$ and $\partial, \partial' \in \mathcal{D}(\mathcal{S}_0^+)$. Moreover,

$$I(\partial\partial') = I(\partial'\partial) = (v_0^0 v_0'^0, (v_{-1}^1 v_0'^0, v_0^1 v_0'^0, v_1^1 v_0'^0)) \quad (88)$$

if $I(\partial) = (v_0^0, (v_{-1}^1, v_0^1, v_1^1))$ and $I(\partial') = (v_0'^0, (0, 0, 0))$.

Proof. Since $\mathbf{u}^{(1/2)} \otimes \mathbf{u}^{(1/2)} \cong \mathbf{u}^{(0)} \oplus \mathbf{u}^{(1)}$, $p_{i,j}$ can be reconstructed from v with help of (84) and the Clebsch-Gordan coefficients. Therefore I is bijective. Clearly, by equation (80) the operator $\partial = \mathrm{id}$ corresponds to $p_{i,j} = \check{C}^{ij}\varepsilon$. From (84) and (116) we conclude that $v_0^0 = -1/[2]\hat{C}_{ij}p_{i,j} = \varepsilon$ and $v_i^1 = 0$ in the case $\partial = \mathrm{id}$. Now we only have to prove (88). Since $v_0'^0 = -1/[2]\hat{C}_{ij}p_{i,j}'$ and $v_k^1 = 0$, we get $p_{i,j}' = \check{C}^{ij}v_0'^0$. Hence $\partial'(a\psi_k^+) = a_{(1)}v_0'^0(a_{(2)})\psi_k^+$ and $\partial\partial'(a\psi_k^+) = a_{(1)}(p_{i,j}v_0'^0)(a_{(2)})\hat{C}_{jk}\psi_i^+$. Equation (86) gives that ∂' commutes with ∂ . Therefore (88) follows from the definition of I . \blacksquare

By the general theory (see e.g. (14.39) in [16]) the equation $\mathrm{ad}_R(f)X_{ij} = f(u_k^i u_l^j)X_{kl}$ holds. Since $(\check{C}^{ij}) \in \mathrm{Mor}(1, u \otimes u)$, it follows that

$$\mathrm{ad}_R(f)(X_{ij} + q^{1/2}/\hat{q} \cdot \check{C}^{ij}) = f(u_k^i u_l^j)(X_{kl} + q^{1/2}/\hat{q} \cdot \check{C}^{kl}) \quad (89)$$

for $i, j = 1, 2$. Let us define the functionals D_{ij}^0 and D_{ij}^1 , $i, j = 1, 2$, by

$$D_{ij}^0 = -1/[2]\check{C}^{ij}\hat{C}_{kl}(X_{kl} + q^{1/2}\check{C}^{kl}/\hat{q}), \quad D_{ij}^1 = X_{ij} + q^{1/2}\check{C}^{ij}/\hat{q} - D_{ij}^0. \quad (90)$$

Then obviously $\mathrm{ad}_R(f)D_{ij}^m = f(u_k^i u_l^j)D_{kl}^m$ for $m = 0, 1$. Hence the left-invariant differential operators ∂_m , defined by

$$\partial_m(a\psi_i^+) := a_{(1)}D_{kj}^m(a_{(2)})\hat{C}_{ji}\psi_k^+, \quad m = 0, 1, \quad (91)$$

are invariant.

Theorem 7.2. Recall the definitions (90) and (91). The algebra of invariant differential operators $\mathcal{D}(\mathcal{S}_0^+)$ on \mathcal{S}_0^+ is isomorphic to the free commutative unital algebra $\tilde{\mathcal{D}}(\mathcal{S}_0^+)$ generated by ∂_0 and ∂_1 and the relation

$$(\partial_1)^2 + q^{-1}\hat{q}\partial_0\partial_1 - q^{-2}\partial_0^2 = -q^{-1}/\hat{q}^2 \cdot \text{id}. \quad (92)$$

Proof. First we compute $\theta^2 a\psi_i^+$ in two ways which will lead to (92). Of course, $\theta^2 = g(\theta, \theta) = -[2]c_1/\hat{q}^2$. On the other hand, $\theta a = da + a\theta = a_{(1)}(X_{kl} + q^{1/2}\check{C}^{kl}/\hat{q})(a_{(2)})\theta_{kl}$. Comparing $(\theta(\theta a))\psi_i^+$ and $(\theta\theta)a\psi_i^+$, this and equations (57) give

$$\begin{aligned} a_{(1)}((X_{rs} + q^{1/2}\check{C}^{rs}/\hat{q})(X_{kl} + q^{1/2}\check{C}^{kl}/\hat{q}))(a_{(2)})\theta_{rs}\theta_{kl}\psi_i^+ &= \\ = q^{1/2}[2]c_1\hat{C}_{sm}\hat{C}_{ni}\hat{R}_{kl}^{mn}a_{(1)}((X_{rs} + q^{1/2}\check{C}^{rs}/\hat{q})(X_{kl} + q^{1/2}\check{C}^{kl}/\hat{q}))(a_{(2)})\psi_r^+ & \\ = -[2]c_1/\hat{q}^2 \cdot a\psi_i^+. \end{aligned} \quad (93)$$

Multiplying and setting $\hat{R}_{kl}^{mn} = q^{1/2}\delta_k^m\delta_l^n + q^{-1/2}\check{C}^{mn}\hat{C}_{kl}$ the latter becomes

$$\begin{aligned} a_{(1)}(X_{rs}\hat{C}_{sk}X_{kl}\hat{C}_{li} + q^{-1/2}X_{rl}\hat{C}_{li} & \\ + q^{-1}X_{rl}\hat{C}_{li}\hat{C}_{mn}X_{mn} + q^{-1/2}/\hat{q} \cdot \hat{C}_{mn}X_{mn}\delta_i^r)(a_{(2)})\psi_r^+ &= 0. \end{aligned} \quad (*)$$

Set $\tilde{\partial}_0(a\psi_i^+) := a_{(1)}\hat{C}_{rs}X_{rs}(a_{(2)})\psi_i^+$ and $\tilde{\partial}_1(a\psi_i^+) := a_{(1)}X_{kj}(a_{(2)})\hat{C}_{ji}\psi_k^+$. Then

$$\begin{aligned} \partial_0 &= -1/[2]\tilde{\partial}_0 + q^{1/2}/\hat{q} \cdot \text{id}, & \partial_1 &= \tilde{\partial}_1 + 1/[2]\tilde{\partial}_0, \\ \tilde{\partial}_0 &= -[2]\partial_0 + q^{1/2}[2]/\hat{q} \cdot \text{id}, & \tilde{\partial}_1 &= \partial_1 + \partial_0 - q^{1/2}/\hat{q} \cdot \text{id}. \end{aligned} \quad (94)$$

Moreover, (*) is equivalent to

$$(\tilde{\partial}_1)^2 + q^{-1/2}\tilde{\partial}_1 + q^{-1}\tilde{\partial}_1\tilde{\partial}_0 + q^{-1/2}/\hat{q} \cdot \tilde{\partial}_0 = 0. \quad (95)$$

From this equation (92) easily follows.

In [14] was proved that the locally finite part $\mathcal{F}(\mathcal{A}^\circ)$, defined by

$$\mathcal{F}(\mathcal{A}^\circ) := \{f \in \mathcal{A}^\circ \mid \dim \text{ad}_R(\mathcal{A}^\circ)f < \infty\}, \quad (96)$$

is isomorphic to the vector space

$$\mathcal{F}(\mathcal{A}^\circ) = \bigoplus_{n \in 1/2\mathbb{N}_0} \ell(\mathbf{u}^{(n)}) \oplus \varepsilon_- \bigoplus_{n \in 1/2\mathbb{N}_0} \ell(\mathbf{u}^{(n)}). \quad (97)$$

Because of $da = \theta a - a\theta$, $a \in \mathcal{A}$, we get $X_{ij} = q^{1/2}/\hat{q}(\check{C}^{kl}f_{ij}^{kl} - \check{C}^{ij}\varepsilon)$. Therefore

$$\begin{aligned} \Delta(X_{ij} + q^{1/2}\check{C}^{ij}/\hat{q}\varepsilon) &= X_{kl} \otimes f_{ij}^{kl} + \varepsilon \otimes X_{ij} + q^{1/2}\check{C}^{ij}/\hat{q}\varepsilon \otimes \varepsilon \\ &= (X_{kl} + q^{1/2}\check{C}^{kl}/\hat{q}\varepsilon) \otimes f_{ij}^{kl}. \end{aligned}$$

This together with Theorem 4.1 in [14] (or direct computation) gives that $\varepsilon_{\pm}\ell(\mathbf{u}^{1/2}) = \text{Lin}\{X_{ij} + q^{1/2}\check{C}^{ij}/\hat{q}\varepsilon\}$ for the $4D_{\pm}$ -calculus. Then Proposition 2.6 in [14] gives that

$$\langle \mathcal{X}_{\pm} \rangle = \bigoplus_{n \in 1/2\mathbb{N}_0} \varepsilon_{\pm}^n \ell(\mathbf{u}^{(n)}). \quad (98)$$

Moreover, $\text{ad}_R(\mathcal{A}^{\circ})(\varepsilon_{\pm}\ell(\mathbf{u}^{(n)})) \subset \varepsilon_{\pm}\ell(\mathbf{u}^{(n)})$. Therefore $p_{i,j}$ can be written as a finite sum of functionals $p_{i,j}^{(n)}$, $n \in 1/2\mathbb{N}_0$, and each differential operator corresponding to the functionals $p_{i,j}^{(n)}$ for a fixed $n \in 1/2\mathbb{N}_0$ is invariant.

For $n \in 1/2\mathbb{N}_0$ fix a nonzero matrix $F(n) = (F(n)_j^i)$, $i, j = -n, -n+1, \dots, n$, such that $F(n) \in \text{Mor}(\mathbf{u}^{(n)}, (\mathbf{u}^{(n)})^{\circ})$. Since $\mathbf{u}^{(n)}$ is an irreducible corepresentation, $F(n)$ is invertible and unique up to a complex factor. Now let us define the mappings $\tilde{\ell} : \mathbf{u}^{(n)} \rightarrow \mathcal{A}^{\circ}$ by $\tilde{\ell}(\mathbf{u}_{ij}^{(n)}) = \ell(\mathbf{u}_{ik}^{(n)})F(n)^{-1j}_k$. Because of $\text{ad}_R(f)\ell(\mathbf{u}_{ij}^{(n)}) = f(\mathbf{u}_{ik}^{(n)}S(\mathbf{u}_{lj}^{(n)}))\ell(\mathbf{u}_{kl}^{(n)})$ we conclude that

$$\text{ad}_R(f)\tilde{\ell}(\mathbf{u}_{ij}^{(n)}) = f(\mathbf{u}_{ik}^{(n)}\mathbf{u}_{jl}^{(n)})\tilde{\ell}(\mathbf{u}_{kl}^{(n)}). \quad (99)$$

From (85) with $l_1 = l_2 = n$ we obtain the formula and hence

$$\text{ad}_R(f)\tilde{\ell}(C_q^{-1}(n, n, t; i, j, r)\mathbf{u}_{ij}^{(n)}) = f(C_q^{-1}(n, n, t; k, l, s)\mathbf{u}_{rs}^{(t)})\tilde{\ell}(\mathbf{u}_{kl}^{(n)}) \quad (100)$$

for all r, n and all $t = 0, 1, \dots, 2n$. This proves that for each $n \in 1/2\mathbb{N}_0$ there exists a unique 1-dimensional complex subspace $V^{(n),1}$ of $\mathbf{u}^{(n)}$ and a unique 3-dimensional one $V^{(n),3}$ with basis $\{v_0^{(n),1}\}$ and $\{v_i^{(n),3} \mid i = -1, 0, 1\}$, respectively, such that

$$\text{ad}_R(f)\tilde{\ell}(v_i^{(n),2m+1}) = f(\mathbf{u}_{ij}^{(m)})\tilde{\ell}(v_j^{(n),2m+1}), \quad m = 0, 1, i = -m, \dots, m. \quad (101)$$

Moreover, these bases are unique up to a nonzero complex factor.

Setting $m = 0$, (101) implies that $\tilde{\ell}(v_0^{(n),1})$ is a central element in \mathcal{A}° . In particular, $v_0^{(1/2),1}$ can be chosen in such a way that $\tilde{\ell}(v_0^{(1/2),1})$ becomes the quantum Casimir element \mathcal{Z}_q of \mathcal{A}° . Recall that there is no nontrivial polynomial function $p(\cdot)$ with complex coefficients such that $p(\varepsilon_{\pm}\mathcal{Z}_q) \equiv 0$. Therefore, since $\tilde{\ell}(\mathbf{u}^{(n_1)})\tilde{\ell}(\mathbf{u}^{(n_2)}) = \tilde{\ell}(\mathbf{u}^{(n_1)} \otimes \mathbf{u}^{(n_2)})$ for $n_1, n_2 \in 1/2\mathbb{N}_0$, $\varepsilon_{\pm}^{2n}\tilde{\ell}(v_0^{(n),1})$ is a polynomial function of the element $\varepsilon_{\pm}\mathcal{Z}_q$ of degree $2n$. Further, $\mathcal{Z}_q \in \tilde{\ell}(\mathbf{u}^{(1/2)})$ implies that

$\mathcal{Z}_q^m \in \tilde{\mathcal{L}}((\mathbf{u}^{(1/2)})^{\otimes m})$, $m \in \mathbb{N}$. We obtain

$$\dim \bigoplus_{m=0}^n \varepsilon_{\pm}^{m+1} \mathcal{Z}_q^m \tilde{\mathcal{L}}(V^{(1/2),3}) = 3n + 3, \quad (102)$$

$$\varepsilon_{\pm}^{m+1} \mathcal{Z}_q^m \tilde{\mathcal{L}}(V^{(1/2),3}) \subset \bigoplus_{\substack{l=1 \\ m+1-l \in 2\mathbb{Z}}}^{m+1} \varepsilon_{\pm}^{m+1} \tilde{\mathcal{L}}(V^{(l/2),3}) = \bigoplus_{\substack{l=1 \\ m+1-l \in 2\mathbb{Z}}}^{m+1} \varepsilon_{\pm}^l \tilde{\mathcal{L}}(V^{(l/2),3}), \quad (103)$$

$$\dim \bigoplus_{l=1}^{n+1} \varepsilon_{\pm}^l \tilde{\mathcal{L}}(V^{(l/2),3}) = 3n + 3 \quad (104)$$

for all $n \in \mathbb{N}$. This means that for all $n \in \mathbb{N}$ there exists a polynomial $P_n(\cdot)$ of degree $n - 1$ such that $\varepsilon_{\pm}^n \tilde{\mathcal{L}}(v_i^{(n/2),3}) = P_n(\varepsilon_{\pm} \mathcal{Z}_q) \varepsilon_{\pm} \tilde{\mathcal{L}}(v_i^{(1/2),3})$, $i = -1, 0, 1$. Denoting $I^{-1}((0, (\varepsilon_{\pm}^n \tilde{\mathcal{L}}(v_{-1}^{(n/2),3}), \varepsilon_{\pm}^n \tilde{\mathcal{L}}(v_0^{(n/2),3}), \varepsilon_{\pm}^n \tilde{\mathcal{L}}(v_1^{(n/2),3}))))$ by ∂ , the latter equation and Lemma 7.1 give $\partial = P_n(\partial_0) \partial_1$. Hence $\partial \in \tilde{\mathcal{D}}(\mathcal{S}_0^+)$. ■

Let (e_1, e_{-1}) be a fixed basis of \mathbb{C}^2 .

Corollary 7.3. *The algebra of invariant differential operators $\mathcal{D}(\mathcal{S}_0)$ on \mathcal{S}_0 is isomorphic to the algebra $\tilde{\mathcal{D}}(\mathcal{S}_0) := \mathbb{C}^2 \otimes \tilde{\mathcal{D}}(\mathcal{S}_0^+) \otimes \mathbb{C}^2$ with multiplication*

$$(\alpha \otimes \partial \otimes \beta)(\alpha' \otimes \partial' \otimes \beta') = \delta_{\alpha'}^{\beta} \alpha \otimes \partial \partial' \otimes \beta, \quad (105)$$

where $\alpha, \alpha', \beta, \beta' \in \{e_1, e_{-1}\}$, $\partial, \partial' \in \tilde{\mathcal{D}}(\mathcal{S}_0^+)$.

Proof. The mappings $\partial_{+-}, \partial_{-+} : \mathcal{S}_0 \rightarrow \mathcal{S}_0$, defined by $\partial_{+-}(\psi_i^+) = \partial_{-+}(\psi_i^-) = 0$, $\partial_{+-}(\psi_i^-) = \psi_i^+$, $\partial_{-+}(\psi_i^+) = \psi_i^-$, are elements of $\mathcal{D}(\mathcal{S}_0)$. One should identify the mapping $e_1 \otimes \text{id} \otimes e_{-1} \in \tilde{\mathcal{D}}(\mathcal{S}_0)$ with the mapping ∂_{+-} and $e_{-1} \otimes \text{id} \otimes e_1$ with the mapping ∂_{-+} , respectively. Then the element $e_1 \otimes \partial' \otimes e_1 \in \tilde{\mathcal{D}}(\mathcal{S}_0)$ can be identified with $\partial' \in \tilde{\mathcal{D}}(\mathcal{S}_0^+)$. ■

By the defining equation (38) and by Theorem 6.7 we are able to determine explicitly the left connection $\nabla_{\mathcal{S}}^*$ dual to $\nabla_{\mathcal{S}}$. Surprisingly the computations result in the formula $\nabla_{\mathcal{S}}^*(\psi) = \theta \otimes_{\mathcal{A}} \psi - \tau^*(\psi \otimes_{\mathcal{A}} \theta)$ for $\psi \in \mathcal{S}_0$, where

$$\begin{aligned} \tau^*(\psi_m^+ \otimes_{\mathcal{A}} \theta_{jk}) &= \nu \bar{\gamma} q^{j+k} \hat{R}_{lk}^{st} \hat{R}_{mj}^{-\eta_i' rl} \theta_{rs} \otimes_{\mathcal{A}} \psi_t^+, \\ \tau^*(\psi_m^- \otimes_{\mathcal{A}} \theta_{jk}) &= \bar{\gamma} q^{j+k} \hat{R}_{lk}^{-\eta_i st} \hat{R}_{mj}^{-1rl} \theta_{rs} \otimes_{\mathcal{A}} \psi_t^-, \end{aligned} \quad \text{for } q \in \mathbb{R}, \quad (106)$$

$$\begin{aligned} \tau^*(\psi_m^+ \otimes_{\mathcal{A}} \theta_{jk}) &= q^{-6} \bar{\gamma} q^{j+k} \hat{R}_{lk}^{st} \hat{R}_{mj}^{\eta_i rl} \theta_{rs} \otimes_{\mathcal{A}} \psi_t^+, \\ \tau^*(\psi_m^- \otimes_{\mathcal{A}} \theta_{jk}) &= \nu q^{-6} \bar{\gamma} q^{j+k} \hat{R}_{lk}^{\eta_i st} \hat{R}_{mj}^{-1rl} \theta_{rs} \otimes_{\mathcal{A}} \psi_t^-, \end{aligned} \quad \text{for } |q| = 1. \quad (107)$$

Comparing these formulas with (76) it turns out that the left connection $\nabla_{\mathcal{S}}^*$ is a linear left connection on the spinor module \mathcal{S}_0 . However, for $q \in \mathbb{R}$, $\nabla_{\mathcal{S}}$ and

∇_S^* are compatible with the same linear connection on $\text{Cl}(I, \sigma, g)$ if and only if $\eta_i = -\eta'_i$, that is for $i = 2$ and $i = 3$.

Corollary 7.4. *Let ∇_S be a linear connection on the spinor module \mathcal{S}_0 . Then the corresponding operators D and $\nabla^*\nabla_S$ are elements of $\mathcal{D}(\mathcal{S}_0)$. Moreover, both D and $\nabla^*\nabla_S$ are invariant first order differential operators on \mathcal{S}_0 .*

Proof. The assertion follows at once if we have shown that the formulas

$$D = e_{-1} \otimes (-q^{1/2}\partial_1 + q^{-3/2}\partial_0 - q^{7/2-3/2\eta_i}\gamma/\hat{q}) \otimes e_1 \quad (108)$$

$$+ e_1 \otimes (-q^{1/2}[2]c_1\partial_1 - q^{1/2}[2]c_1\partial_0 + q^{5/2-3/2\eta'_i}[2]c_1\nu\gamma/\hat{q}) \otimes e_{-1},$$

$$\nabla^*\nabla_S = e_1 \otimes (\alpha_1\partial_1 + \alpha_0\partial_0 + \alpha_2) \otimes e_1 \quad (109)$$

$$+ e_{-1} \otimes (\beta_1\partial_1 + \beta_0\partial_0 + \beta_2) \otimes e_{-1}$$

hold, where

$$\begin{aligned} \alpha_1 &= q^{7/2}c_1(\gamma(1 - \eta_i)/2 + \nu\bar{\gamma}(1 + \eta'_i)/2), \\ \alpha_0 &= q^{5/2}c_1/\hat{q} \cdot ((q^{3/2-\eta_i/2} + q^{-3/2+\eta_i/2})\gamma + (q^{3/2+\eta'_i/2} + q^{-3/2-\eta'_i/2})\nu\bar{\gamma}), \\ \alpha_2 &= -c_1/\hat{q}^2 \cdot ([2] + q^6(q^{1+|\eta_i+\eta'_i|/2} + q^{-1-|\eta_i+\eta'_i|/2})\nu\gamma\bar{\gamma}), \\ \beta_1 &= q^{7/2}c_1(\bar{\gamma}(1 - \eta_i)/2 + \nu\gamma(1 + \eta'_i)/2), \\ \beta_0 &= q^{5/2}c_1/\hat{q} \cdot ((q^{3/2-\eta_i/2} + q^{-3/2+\eta_i/2})\bar{\gamma} + (q^{3/2+\eta'_i/2} + q^{-3/2-\eta'_i/2})\nu\gamma), \\ \beta_2 &= -c_1/\hat{q}^2 \cdot ([2] + q^6(q^{1+|\eta_i+\eta'_i|/2} + q^{-1-|\eta_i+\eta'_i|/2})\nu\gamma\bar{\gamma}) \end{aligned}$$

for $q \in \mathbb{R}$ and

$$\begin{aligned} \alpha_1 &= q^{1/2}c_1(1 - \eta_i)/2 \cdot (q^3\gamma + q^{-3}\bar{\gamma}), \\ \alpha_0 &= c_1/\hat{q} \cdot (q^{5/2}(q^{3/2-\eta_i/2} + q^{-3/2+\eta_i/2})\gamma + q^{-7/2}(q^{3/2-\eta_i/2} + q^{-3/2+\eta_i/2})\bar{\gamma}), \\ \alpha_2 &= -c_1[2]/\hat{q}^2 \cdot (1 + \gamma\bar{\gamma}), \\ \beta_1 &= q^{1/2}c_1(1 + \eta'_i)/2 \cdot (q^3\nu\gamma + q^{-3}\nu\bar{\gamma}), \\ \beta_0 &= c_1/\hat{q} \cdot (q^{5/2}(q^{3/2+\eta'_i/2} + q^{-3/2-\eta'_i/2})\nu\gamma + q^{-7/2}(q^{3/2+\eta'_i/2} + q^{-3/2-\eta'_i/2})\nu\bar{\gamma}), \\ \beta_2 &= -c_1[2]/\hat{q}^2 \cdot (1 + \gamma\bar{\gamma}) \end{aligned}$$

for $|q| = 1$.

By equation (20) we obtain $D(a\psi) = a_{(1)}X_{kl}(a_{(2)})\theta_{kl}\psi + aD\psi$ for $a \in \mathcal{A}$, $\psi \in \mathcal{S}_0$. Inserting (78) and (57) and using the mappings $\tilde{\partial}_0$ and $\tilde{\partial}_1$ defined before

(94) this gives

$$\begin{aligned} D(a\psi_m^+) &= a_{(1)}X_{kl}(a_{(2)})(-q^{1/2}\hat{C}_{lm}\psi_k^- - q^{-1/2}\hat{C}_{kl}\psi_m^-) + \frac{q^{-1} - q^{7/2-3/2\eta_i}\gamma}{\hat{q}}a\psi_m^- \\ &= \left(e_{-1} \otimes \left(-q^{1/2}\tilde{\partial}_1 - q^{-1/2}\tilde{\partial}_0 + \frac{q^{-1} - q^{7/2-3/2\eta_i}\gamma}{\hat{q}} \right) \otimes e_1 \right) a\psi_m^+. \end{aligned}$$

Similar computations for $D(a\psi_m^-)$ and the transformation formulas (94) yield (108).

For the computation of (109) we use (43). First, the equation $g(\theta \otimes_{\mathcal{A}} \theta a) - g(\theta \otimes_{\mathcal{A}} \theta)a = 0$ gives

$$a_{(1)}(X_{kl}X_{mn})(a_{(2)})g(\theta_{kl}, \theta_{mn}) + 2a_{(1)}X_{kl}(a_{(2)})g(\theta_{kl}, \theta) = 0,$$

and hence $a_{(1)}(X_{kl}X_{mn})(a_{(2)})g(\theta_{kl}, \theta_{mn}) = -2q^{-1/2}c_1/\hat{q}a_{(1)}\hat{C}_{kl}X_{kl}(a_{(2)})$. Consider the case when $q \in \mathbb{R}$. From (77) we derive that

$$\begin{aligned} a_{(1)}X_{kl}(a_{(2)})(g \otimes \text{id})(\theta_{kl} \otimes_{\mathcal{A}} (\nabla_S + \nabla_S^*)(\psi_m^+)) &= \\ &= (c_1/\hat{q} \cdot (2q^{-1/2} - q^{2+\eta_i/2}\gamma - q^{2-\eta'_i/2}\nu\bar{\gamma})\tilde{\partial}_0 \\ &\quad + q^{7/2}c_1((1-\eta_i)\gamma/2 + (1+\eta'_i)\nu\bar{\gamma}/2)\tilde{\partial}_1)(a\psi_m^+) \end{aligned}$$

and

$$\begin{aligned} (g \otimes \text{id})(\text{id} \otimes \nabla_S^*)\nabla_S(\psi_m^+) &= -c_1/\hat{q}^2 \cdot ([2] - q^3(q^{3/2-\eta_i/2} + q^{-3/2+\eta_i/2})\gamma \\ &\quad - q^3(q^{3/2+\eta'_i/2} + q^{-3/2-\eta'_i/2})\nu\bar{\gamma} + q^6(q^{1+|\eta_i+\eta'_i|/2} + q^{-1-|\eta_i+\eta'_i|/2})\nu\gamma\bar{\gamma}). \end{aligned}$$

These together with (94) give the first summand for the connection Laplacian. The other one, corresponding to $\nabla^*\nabla_S \upharpoonright \mathcal{S}_0^-$, and the formula for $|q| = 1$ can be determined similarly. \blacksquare

Let $q \in \mathbb{R}$. Suppose that ∇_S is one of the linear left connections on \mathcal{S}_0 from Theorem 6.7 such that $\nu = 1$, $\bar{\gamma} = \gamma$ and $\eta_i = \eta'_i$ (that is $i = 1$ or $i = 4$). Then D is symmetric by Theorem 6.7.(ii), but ∇_S and ∇_S^* are not compatible with the same left connection on $\text{Cl}(\Gamma, \sigma, g)$ (see the considerations before Corollary 7.4). For these connections one can formulate a modification of Bochner's theorem.

Theorem 7.5. *Let ∇_S be as above and let D and $\nabla^*\nabla_S$ be the corresponding Dirac operator and connection Laplacian, respectively. Then the operator*

$$(q+1)D^2 - q^{1/2-3/2\eta_i}[2]\nabla^*\nabla_S = \eta_i q^{-\eta_i-1}[2]c_1 \frac{q^3-1}{\hat{q}} \frac{q^6\gamma^2-1}{\hat{q}} \text{id} \quad (110)$$

is an invariant differential operator of order zero.

Proof. Using equations (108) and (92) we obtain that $D^2 = e_1 \otimes x \otimes e_1 + e_{-1} \otimes x \otimes e_{-1}$ (in the notation of Corollary 7.3), where x denotes the operator

$$[2]c_1/\hat{q}((q^{4-3/2\eta_i} - q^{3-3/2\eta'_i}\nu)\gamma\partial_1 + (q^{4-3/2\eta_i} + q^{1-3/2\eta'_i}\nu)\gamma\partial_0 - (1 + q^{6-3/2\eta_i-3/2\eta'_i}\nu\gamma^2)/\hat{q}). \quad (111)$$

Setting $\nu = 1$ and $\eta'_i = \eta_i$, x becomes

$$[2]c_1/\hat{q}(q^{3-3/2\eta_i}(q-1)\gamma\partial_1 + q^{1-3/2\eta_i}(q^3+1)\gamma\partial_0 - (1 + q^{6-3\eta_i}\gamma^2)/\hat{q}).$$

On the other hand, inserting $\nu = 1$, $\bar{\gamma} = \gamma$ and $\eta'_i = \eta_i$ into equation (109) we directly obtain that $\nabla^*\nabla_S = e_1 \otimes y \otimes e_1 + e_{-1} \otimes y \otimes e_{-1}$, where y denotes the operator

$$q^{7/2}c_1\gamma\partial_1 + q^{1/2}c_1(q+1)(q^3+1)\gamma/\hat{q}\partial_0 - c_1([2] + q^6(q^2 + q^{-2})\gamma^2)/\hat{q}^2.$$

From these formulas (110) immediately follows. ■

Let us compute the eigenvalues of the Dirac operator D corresponding to a linear left connection on the spinor module \mathcal{S}_0 . Since D maps \mathcal{S}_0^η onto $\mathcal{S}_0^{-\eta}$, $\eta \in \{+, -\}$, it suffices to determine the eigenvalues of D^2 on \mathcal{S}_0^+ . Indeed, if $\psi = \psi' \oplus \psi''$, $\psi' \in \mathcal{S}_0^+$, $\psi'' \in \mathcal{S}_0^-$, and $D\psi = \lambda\psi$, $\lambda \in \mathbb{C}$, then $D\psi' = \lambda\psi''$ and $D\psi'' = \lambda\psi'$. Hence $D^2\psi' = \lambda^2\psi'$. Conversely, if $D^2\psi' = \lambda^2\psi'$, $\psi' \in \mathcal{S}_0^+$, $\lambda \in \mathbb{C}^\times$, then $\psi' \pm 1/\lambda^{1/2}D\psi'$ is an eigenvector of D to the eigenvalue $\pm\lambda^{1/2}$.

In all cases, the proof of the above theorem shows that $D^2|_{\mathcal{S}_0^+}$ is of the form $\alpha_1\tilde{\partial}_1 + \alpha_0\tilde{\partial}_0 + \alpha_2\text{id}$, where $\alpha_1, \alpha_0, \alpha_2 \in \mathbb{C}$. Since $\mathcal{D}(\mathcal{S}_0^+)$ is a commutative algebra, $\tilde{\partial}_0$ and $\tilde{\partial}_1$ have common eigenvectors. Our first result will be that the linear hull of these common eigenvectors span \mathcal{S}_0^+ . Indeed, by left-invariance of $\tilde{\partial}_0$ and $\tilde{\partial}_1$ the vector spaces $V_n := \mathbf{u}^{(n)}(\mathcal{S}_0^+)_L$, $n \in \mathbb{N}/2$, are invariant under the action of these differential operators. On the other hand, V_n splits under the right coaction Δ_R into the direct sum $V_n = V_n^+ \oplus V_n^-$, where

$$\begin{aligned} V_n^+ &= \text{Lin}\{\mathbf{u}_r^{(n)}s\psi_i^+C_q(n, 1, n+1/2, s, i, t)\} \quad \text{and} \\ V_n^- &= \text{Lin}\{\mathbf{u}_r^{(n)}s\psi_i^+C_q(n, 1, n-1/2, s, i, t)\} \quad (V_0^- = \{0\}) \end{aligned}$$

are non-isomorphic irreducible bicovariant vector spaces. Since $\tilde{\partial}_0$ and $\tilde{\partial}_1$ are invariant differential operators, by Schurs lemma they act by multiplication with

a scalar on these vector spaces. Because of (115) we have $\mathcal{S}_0^+ = \bigoplus_{n \in \mathbb{N}/2} V_n$, and hence $\tilde{\partial}_0$ and $\tilde{\partial}_1$ are commonly diagonalizable.

For the computation of the eigenvalues of $\tilde{\partial}_0$ and $\tilde{\partial}_1$ on the eigenspaces V_m^\pm , $m \in 1/2\mathbb{N}_0$, we use the canonical embedding of $\mathbf{u}^{(m)}$ into the $2m$ -fold tensor product of the corepresentation $\mathbf{u}^{(1/2)}$. Let $P_{(2m)}$ denote the unique covariant projection of $(\mathbf{u}^{(1/2)})^{\otimes 2m}$ onto its subcoalgebra $\mathbf{u}^{(m)}$. Of course we have $\hat{C}_{i,i+1}P_{(2m)} = P_{(2m)}\check{C}_{i,i+1} = 0$ and $\hat{R}_{i,i+1}P_{(2m)} = P_{(2m)}\hat{R}_{i,i+1} = q^{1/2}P_{(2m)}$ for $1 \leq i < 2m$. Therefore the vector spaces V_m^+ and V_m^- are generated by the elements

$$\begin{aligned} & u_{l_1}^{j_1} u_{l_2}^{j_2} \cdots u_{l_{2m}}^{j_{2m}} \psi_{l_{2m+1}}^+ P_{(2m+1)}^{l_1 l_2 \cdots l_{2m+1}}_{n_1 n_2 \cdots n_{2m+1}} \quad \text{and} \\ & u_{l_1}^{j_1} u_{l_2}^{j_2} \cdots u_{l_{2m}}^{j_{2m}} \psi_{n_{2m+1}}^+ P_{(2m)}^{l_1 l_2 \cdots l_{2m}}_{n_1 n_2 \cdots n_{2m}} \check{C}^{n_{2m} n_{2m+1}}, \end{aligned} \quad (112)$$

respectively.

The definition $da = \theta a - a\theta$, $\theta = \sum_{i,j} q^{1/2} \check{C}^{ij} / \hat{q} \theta_{ij}$, and formula (46) imply that

$$X_{ij}(u_{l_1}^{k_1} u_{l_2}^{k_2} \cdots u_{l_m}^{k_m}) = q^{1/2} / \hat{q} \left((-1)^m \begin{array}{c} k_1 \quad k_2 \quad k_m \quad i \quad j \\ \text{diagram with crossings} \\ l_1 \quad l_2 \quad l_m \end{array} - \begin{array}{c} k_1 \quad k_2 \quad k_m \quad i \quad j \\ \text{diagram with no crossings} \\ l_1 \quad l_2 \quad l_m \end{array} \right)$$

Using the graphical calculus, from this one can easily compute the eigenvalue of $\tilde{\partial}_0$ and $\tilde{\partial}_1$ to the eigenvectors given by (112). We obtain that

$$\begin{aligned} \tilde{\partial}_0 \upharpoonright V_m^+ &= q^{1/2} / \hat{q} \cdot (q + q^{-1} - \nu_0^m q^{2m+1} - \nu_0^m q^{-2m-1}) \text{id} \quad (m \in 1/2\mathbb{N}_0), \\ \tilde{\partial}_0 \upharpoonright V_m^- &= q^{1/2} / \hat{q} \cdot (q + q^{-1} - \nu_0^m q^{2m+1} - \nu_0^m q^{-2m-1}) \text{id} \quad (m \in 1/2\mathbb{N}), \\ \tilde{\partial}_1 \upharpoonright V_m^+ &= q^{1/2} / \hat{q} \cdot (\nu_0^m q^{2m} - 1) \text{id} \quad (m \in 1/2\mathbb{N}_0), \\ \tilde{\partial}_1 \upharpoonright V_m^- &= q^{1/2} / \hat{q} \cdot (\nu_0^m q^{-2m-2} - 1) \text{id} \quad (m \in 1/2\mathbb{N}). \end{aligned}$$

From the transformation formulas (94) we get

$$\begin{aligned} \alpha_1 \partial_1 + \alpha_0 \partial_0 \upharpoonright V_m^+ &= \frac{q^{1/2} \nu_0^m}{[2] \hat{q}} (\alpha_1 (q^{2m-1} - q^{-2m-1}) + \alpha_0 (q^{2m+1} + q^{-2m-1})) \text{id}, \\ \alpha_1 \partial_1 + \alpha_0 \partial_0 \upharpoonright V_m^- &= \frac{q^{1/2} \nu_0^m}{[2] \hat{q}} (\alpha_1 (q^{-2m-3} - q^{2m+1}) + \alpha_0 (q^{2m+1} + q^{-2m-1})) \text{id}. \end{aligned}$$

Setting the correct parameter values for D^2 from (111) we obtain the formulas

$$\begin{aligned} D^2 \lrcorner V_m^+ &= -[2]_{c_1} \frac{q^{9/2+2m-3/2\eta_i} \nu_0^m \gamma - 1}{\hat{q}} \frac{q^{3/2-2m-3/2\eta'_i} \nu_0^m \nu \gamma - 1}{\hat{q}} \text{id}, \\ D^2 \lrcorner V_m^- &= -[2]_{c_1} \frac{q^{5/2-2m-3/2\eta_i} \nu_0^m \gamma - 1}{\hat{q}} \frac{q^{7/2+2m-3/2\eta'_i} \nu_0^m \nu \gamma - 1}{\hat{q}} \text{id}. \end{aligned} \quad (113)$$

A On the quantum group $\text{SL}_q(2)$

Let $\mathcal{O}(\text{SL}_q(2))$ denote the algebra generated by the elements u_j^i , $i, j = 1, 2$, and relations $\hat{R}_{kl}^{ij} u_m^k u_n^l = u_k^i u_l^j \hat{R}_{mn}^{kl}$ and $u_1^1 u_2^2 - q u_1^2 u_2^1 = 1$, where \hat{R} is given by (44). Setting $|u_j^i| = i - j$ it becomes a graded algebra with grading $|\cdot|$. The coproduct Δ and the antipode S , where

$$\Delta u_j^i = u_k^i \otimes u_j^k, \quad S(u_j^i) = \check{C}^{ik} \hat{C}_{lj} u_k^l \quad (114)$$

make $\mathcal{O}(\text{SL}_q(2))$ to a Hopf algebra. Further, $\mathcal{O}(\text{SL}_q(2))$ is a coquasitriangular Hopf algebra with universal r -form \mathbf{r} , such that $\mathbf{r}(u_j^i, u_l^k) = \hat{R}_{jl}^{ki}$.

If q is not a root of unity then the Hopf algebra $\mathcal{O}(\text{SL}_q(2))$ is cosemisimple. More precisely,

$$\mathcal{O}(\text{SL}_q(2)) = \bigoplus_{n \in \mathbb{N}_0/2} \mathbf{u}^{(n)}, \quad (115)$$

where $\mathbf{u}^{(n)}$ is the linear span of the matrix elements of the unique irreducible $2n + 1$ -dimensional corepresentation of $\mathcal{O}(\text{SL}_q(2))$. Let $\{\mathbf{u}_{ij}^{(n)} \mid i, j = -n, -n + 1, \dots, n\}$, $n \in \mathbb{N}_0/2$, denote the basis of $\mathbf{u}^{(n)}$ given in [18]. Then we have $\Delta(\mathbf{u}_{ij}^{(n)}) = \mathbf{u}_{ik}^{(n)} \otimes \mathbf{u}_{kj}^{(n)}$. Particularly,

$$(\mathbf{u}_{ij}^{(1/2)}) = \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix}, \quad (\mathbf{u}_{ij}^{(1)}) = \begin{pmatrix} u_1^1 u_1^1 & q' u_1^1 u_2^1 & u_2^1 u_2^1 \\ q' u_1^1 u_1^2 & 1 + [2] u_2^1 u_1^2 & q' u_2^1 u_2^2 \\ u_1^2 u_1^2 & q' u_1^2 u_2^2 & u_2^2 u_2^2 \end{pmatrix},$$

where $q' = (1 + q^{-2})^{1/2}$. Let $C_q(n_1, n_2, n; i, j, k)$ denote the Clebsch-Gordan coefficients of the tensor product of the corepresentations $\mathbf{u}^{(n_1)}$ and $\mathbf{u}^{(n_2)}$. Recall that

$$\begin{aligned} C_q(\tfrac{1}{2}, \tfrac{1}{2}, 0; i - \tfrac{3}{2}, j - \tfrac{3}{2}, 0) &= -1/[2]^{1/2} \check{C}^{ij} \quad \text{for } i, j = 1, 2, \\ C_q^{-1}(\tfrac{1}{2}, \tfrac{1}{2}, 0; i - \tfrac{3}{2}, j - \tfrac{3}{2}, 0) &= 1/[2]^{1/2} \hat{C}_{ij} \quad \text{for } i, j = 1, 2. \end{aligned} \quad (116)$$

The Hopf algebra $\mathcal{O}(\text{SL}_q(2))$ carries three non-isomorphic $*$ -structures \dagger , \star and \sharp defined by

- $q \in \mathbb{R}; (u_1^1)^\dagger = u_2^2, (u_2^1)^\dagger = -qu_1^2, (u_1^2)^\dagger = -q^{-1}u_2^1, (u_2^2)^\dagger = u_1^1;$
- $q \in \mathbb{R}; (u_1^1)^\star = u_2^2, (u_2^1)^\star = qu_1^2, (u_1^2)^\star = q^{-1}u_2^1, (u_2^2)^\star = u_1^1;$
- $|q| = 1; (u_j^i)^\sharp = u_j^i$ for $i, j = 1, 2.$

The Hopf \ast -algebras corresponding to the above involutions are denoted by $\mathcal{O}(\mathrm{SU}_q(2))$, $\mathcal{O}(\mathrm{SU}_q(1, 1))$ and $\mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$, respectively.

The functionals $\ell(\mathbf{u}_{ij}^{(n)})$ on $\mathcal{O}(\mathrm{SL}_q(2))$, defined by

$$\ell(\mathbf{u}_{ij}^{(n)})(a) := \mathbf{r}(\mathbf{u}_{ik}^{(n)}, a_{(1)})\mathbf{r}(a_{(2)}, \mathbf{u}_{kj}^{(n)}), \quad (117)$$

are called the generalized ℓ -functionals.

Let $U_q(\mathfrak{sl}_2)$ denote the algebra generated by the elements E, F, K and K^{-1} and relations

$$\begin{aligned} KK^{-1} = K^{-1}K = 1, \quad KE = qEK, \quad KF = q^{-1}FK, \\ EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}}. \end{aligned} \quad (118)$$

We denote the unit element in $U_q(\mathfrak{sl}_2)$ by ε_+ . In this paper we consider the central extension $\tilde{U}_q(\mathfrak{sl}_2) = U_q(\mathfrak{sl}_2) \otimes \varepsilon_-$ of $U_q(\mathfrak{sl}_2)$. The algebra $\tilde{U}_q(\mathfrak{sl}_2)$ can be equipped with a Hopf structure such that

$$\begin{aligned} \Delta(E) &= E \otimes K + K^{-1} \otimes E, & \varepsilon(E) &= 0, & S(E) &= -qE, \\ \Delta(F) &= F \otimes K + K^{-1} \otimes F, & \varepsilon(F) &= 0, & S(F) &= -q^{-1}F, \\ \Delta(K) &= K \otimes K, & \varepsilon(K) &= 1, & S(K) &= K^{-1}, \\ \Delta(\varepsilon_-) &= \varepsilon_- \otimes \varepsilon_-, & \varepsilon(\varepsilon_-) &= 1, & S(\varepsilon_-) &= \varepsilon_-. \end{aligned} \quad (119)$$

There exists a dual pairing between the Hopf algebras $\tilde{U}_q(\mathfrak{sl}_2)$ and $\mathcal{O}(\mathrm{SL}_q(2))$ such that for their generators the following formulas hold (the matrix entries for $f \in \tilde{U}_q(\mathfrak{sl}_2)$ are $f(u_j^i)$):

$$E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}, \quad \varepsilon_- = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (120)$$

Hence $\tilde{U}_q(\mathfrak{sl}_2)$ is a Hopf subalgebra of the dual Hopf algebra $\mathcal{O}(\mathrm{SL}_q(2))^\circ$ of $\mathcal{O}(\mathrm{SL}_q(2))$.

The real forms of $\mathcal{O}(\mathrm{SL}_q(2))$ induce \ast -structures on $\tilde{U}_q(\mathfrak{sl}_2)$. They are uniquely given by the pairing of $\mathcal{O}(\mathrm{SL}_q(2))$ and $\tilde{U}_q(\mathfrak{sl}_2)$ and by equation (5). The explicit formulas are

- $E^\dagger = F$, $F^\dagger = E$, $K^\dagger = K$, $\varepsilon_-^\dagger = \varepsilon_-$,
- $E^\star = -F$, $F^\star = -E$, $K^\star = K$, $\varepsilon_-^\star = \varepsilon_-$,
- $E^\sharp = E$, $F^\sharp = F$, $K^\sharp = K$, $\varepsilon_-^\sharp = \varepsilon_-$.

B Graphical calculus

Here we collect the most important equations related to the matrices \hat{R} , \hat{R}^{-1} , \check{C} and \hat{C} in graphical form. We use the symbols

$$\hat{R} = \text{X}, \quad \hat{R}^{-1} = \text{X}, \quad \check{C} = \cup, \quad \hat{C} = \cap, \quad \text{id} = |.$$

From the settings (44) it easily follows that $\hat{R}_{kl}^{ij} = q^{1/2}\delta_k^i\delta_l^j + q^{-1/2}\check{C}^{ij}\hat{C}_{kl}$ and $\hat{C}_{ij}\check{C}^{ij} = -[2]$. This in turn implies the following formulas:

$$\begin{aligned} \text{X} &= q^{1/2} | \quad | + q^{-1/2} \text{X}, & \text{X} &= q^{-1/2} | \quad | + q^{1/2} \text{X}, \\ \text{X} &= q \text{X} - q^{1/2}\hat{q} | \quad |, & \text{X} &= q^{-1} \text{X} + q^{-1/2}\hat{q} | \quad |, \\ \text{X} &= -q^{-3/2} \cup, & \text{X} &= -q^{3/2} \cup, & \bigcirc &= -[2], \\ \text{X} &= q^{-1} | \quad | + q^{-1/2}\hat{q} \text{X} = q | \quad | + q^{-1}\hat{q} \text{X}, \\ \text{X} &= q | \quad | - q^{1/2}\hat{q} \text{X} = q^{-1} | \quad | - q\hat{q} \text{X}. \end{aligned}$$

References

- [1] Bautista, R., Criscuolo, A., Durdević, M., Rosenbaum, M. and Vergara, J.D.: Quantum Clifford algebras from spinor representations. J. Math. Phys. **37**, 5747–5775 (1996)
- [2] Bresser, K., Dimakis, A., Müller-Hoissen, F. and Sitarz, A.: Noncommutative geometry of finite groups. J. Phys. A. **29**, 2705–2735 (1996)
- [3] Cartan, E.: *The theory of spinors*. Dover, New York (1966)
- [4] Connes, A.: *Non-commutative geometry*. Academic Press, San Diego (1995)

- [5] Crumeyrolle, A.: *Orthogonal and symplectic Clifford algebras*. Kluwer Academic Publishers, Dordrecht (1990)
- [6] Cuntz, J. and Quillen, D.: Algebra extensions and nonsingularity. *J. Amer. Math. Soc.* **8**(2), 251–289 (1995)
- [7] Dąbrowski, L., Hajac, P., Landi, G. and Siniscalco, P.: Metrics and pairs of left and right connections on bimodules. *J. Math. Phys.* **37**(9), 4635–4646 (1996)
- [8] Doi, Y.: Unifying Hopf modules. *J. Algebra.* **153**, 373–385 (1992)
- [9] Dubois-Violette, M., Madore, J., Masson, T. and Mourad, J.: Linear connections on the quantum plane. *Lett. Math. Phys.* **35**(4), 351–358 (1995)
- [10] Faddeev, L.D., Reshetikhin, N.Yu. and Takhtajan, L.A.: Quantization of Lie groups and Lie algebras. *Algebra and Analysis.* **1**, 178–206 (1987)
- [11] Heckenberger, I.: Classification of left-covariant differential calculi on the quantum group $SL_q(2)$. *J. Algebra*, in press. Preprint math.QA/0006211.
- [12] Heckenberger, I.: Hodge and Laplace-Beltrami operators for bicovariant differential calculi on quantum groups. *Compositio Mathematica*, in press. Preprint math.QA/9902130.
- [13] Heckenberger, I. and Schmüdgen, K.: Levi-Civita connections on the quantum groups $SL_q(N)$, $O_q(N)$ and $Sp_q(N)$. *Commun. Math. Phys.* **185**, 177–196 (1997)
- [14] Heckenberger, I. and Schmüdgen, K.: Classification of bicovariant differential calculi on the quantum groups $SL_q(n+1)$ and $Sp_q(2n)$. *J. reine angew. Math.* **502**, 141–162 (1998)
- [15] Heckenberger, I. and Schüler, A.: De Rham cohomology and Hodge decomposition for quantum groups. Preprint NTZ 10/2000.
- [16] Klimyk, A. and Schmüdgen, K.: *Quantum groups and their representations*. Springer-Verlag, Heidelberg (1997)
- [17] Lawson, H.B. and Michelsohn, M.-L.: *Spin geometry*. Princeton mathematical series. Princeton University Press, Princeton/New Jersey (1989)

- [18] Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M. and Ueno, K.: Representations of the quantum group $SU_q(2)$ and the little q -Jacobi polynomials. J. Funct. Anal. **99**, 357–386 (1991)
- [19] Mourad, J.: Linear connections in non-commutative geometry. Class. Quantum Grav. **12**, 965–974 (1995)
- [20] Sweedler, M.E.: *Hopf algebras*. Benjamin, New York (1969)
- [21] Woronowicz, S.L.: Differential calculus on compact matrix pseudogroups (quantum groups). Commun. Math. Phys. **122**(1), 125–170 (1989)